1. $|Z_{12}| = 12; |U(10)| = 4; |U(12)| = 4; |U(20)| = 8; |D_4| = 8.$
   In Z_{12}, $|0| = 1; |1| = |5| = |7| = |11| = 12; |2| = |10| = 6; |3| = |9| = 4; |4| = |8| = 3; |6| = 2.$
   In $U(10)$, $|1| = 1;\ |3| = |7| = 4;\ |9| = 2.$
   In $U(20)$, $|1| = 1;\ |3| = |7| = |13| = |17| = 4;\ |9| = |11| = |19| = 2.$
   In D_4, $|R_0| = 1;\ |R_{90}| = |R_{270}| = 4;$
   $|R_{180}| = |H| = |V| = |D| = |D'| = 2.$
   In each case, notice that the order of the element divides the order of the group.

6. a. $|6| = 2, |2| = 6, |8| = 3;$ b. $|3| = 4, |8| = 5, |11| = 12$;
   c. $|5| = 12, |4| = 3, |9| = 4.$ In each case $|a + b|$ divides $\text{lcm}(|a|, |b|)$.

19. Suppose that $m < n$ and $a^m = a^n$. Then $e = a^n a^{-m} = a^{n-m}$. This contradicts the assumption that a has infinite order.

53. By induction we will prove that any positive integer n we have

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}^n = \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix}.$$

The $n = 1$ case is true by definition. Now assume

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}^k = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}.$$

Then

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}^{k+1} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}^k \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} =$$

$$\begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & k+1 \\ 0 & 1 \end{bmatrix}.$$

So, when the entries are from $\mathbf{R}$, $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ has infinite order.
When the entries are from Z_p, the order is p.

68. Say $\det A = 2^m$ and $\det B = 2^n$. Then $\det(AB) = 2^{m+n}$ and $\det A^{-1} = 2^{-m}$.