Generating Sets for Groups Solutions:

① Show that the subgroup of $\mathbb{Z}$ generated by two non-zero integers a, b is the cyclic subgroup $\langle \gcd(a,b) \rangle$.

Proof: Let $d = \gcd(a,b)$. Then $a = dr$ and $b = ds$ for some $r, s \in \mathbb{Z}$. Thus given $x = am + bn \in \langle a, b \rangle$ we have:

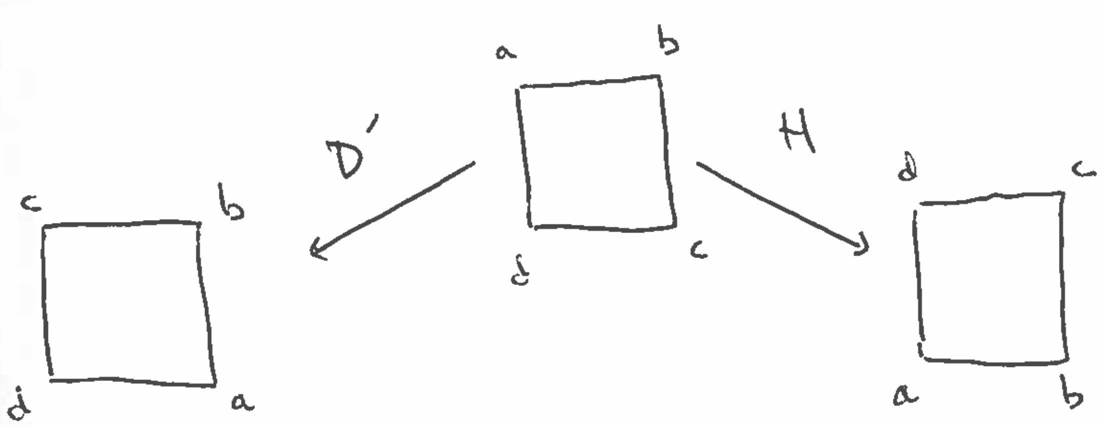
$$x = am + bn = drm + dsn$$
$$= d(rm + sn)$$

Thus $x \in \langle d \rangle$ and so $\langle a,b \rangle \subseteq \langle d \rangle$. Conversely, there exist $p, q \in \mathbb{Z}$ such that $ap + bq = d$. So given $x = dm \in \langle d \rangle$ we have:

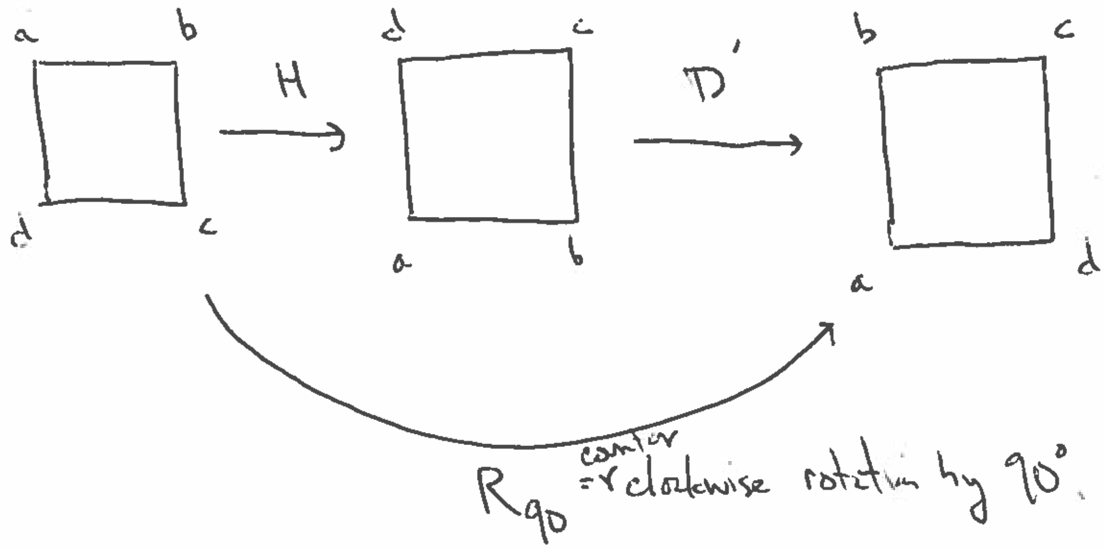
$$x = dm = (ap + bq)m$$
$$= apm + bqm$$
$$= a(pm) + b(qm)$$

Thus $x \in \langle a,b \rangle$ and so $\langle d \rangle \subseteq \langle a,b \rangle$. Therefore $\langle a,b \rangle = \langle d \rangle$ ∎

② Show that D_4 is generated by:



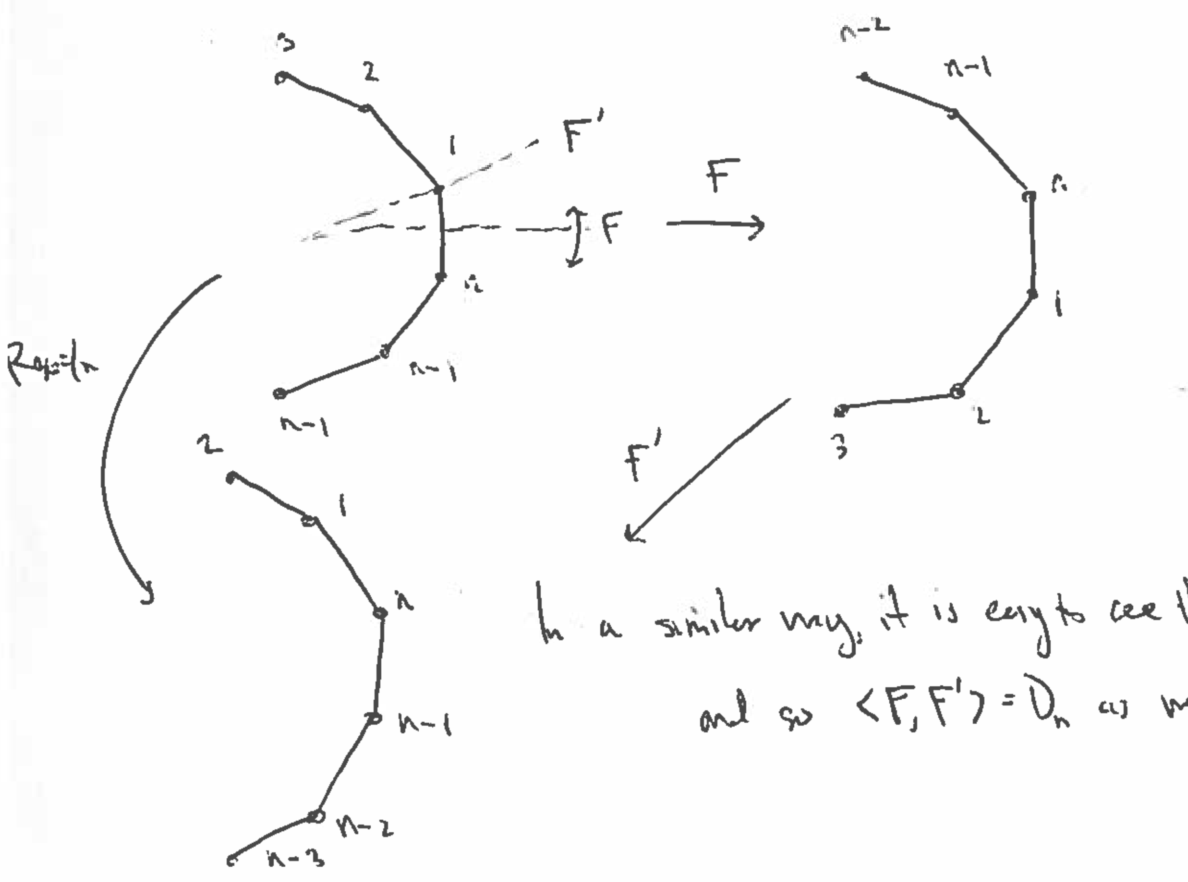
Proof: We compute that $D'H = R_{90}$:



We have already observed that $\langle H, R_{90} \rangle = D_4$ and so $\langle H, D' \rangle = D_4$ as well. ∎

Generalization: find two reflections that generate D_n.

Proof: Let F and F' be reflections of an n-gon whose axes of reflection intersect in an angle of π/n. Then $F'F = R_{360/n}$ as for $n=4$:



In a similar way, it is easy to see that $\langle F, R_{360/n} \rangle = D_n$ and so $\langle F, F' \rangle = D_n$ as well. ∎