Exam 3 Key:

① How many cyclic subgroups of order 12 does $\mathbb{Z}_{40} \oplus \mathbb{Z}_{30}$ contain?

First we count # of elements of order 12:

$|(a,b)| = 12 \iff \text{lcm}(|a|,|b|) = 12 \Rightarrow |a| \mid 12$ so $|a| = 1,2,3,4,6$ or 12

but $|a| \mid 40$ as well, so $|a| = 1,2,4$

If $|a| = 4$ then $|b| = 3$ or 6

$\varphi(4) = 2$ possibilities for a

$\varphi(3) = 2$, $\varphi(6) = 2$

$2+2 = 4$ possibilities for b

$2 \cdot 4 = 8$ elements

If $|a| = 1$ or 2 then $|(a,b)| \neq 12$ as $12 \nmid 30$ so $\nexists\, b \in \mathbb{Z}_{30}$ with $|b| = 12$

So $\exists$ 8 elements in $\mathbb{Z}_{40} \oplus \mathbb{Z}_{30}$ of order 12, as a cyclic subgroup of order 12 contains $\varphi(12) = \varphi(3)\varphi(4) = 2 \cdot 2 = 4$ generators,

$\Rightarrow \exists\ 8/4 = 2$ subgroups of order 12

$\langle(10,10)\rangle$ & $\langle(10,5)\rangle$ are the two subgroups

② $\mathbb{Z}_{18} \oplus \mathbb{Z}_2$ is <u>not</u> isomorphic to $\mathbb{Z}_6 \oplus \mathbb{Z}_6$.

Then element $(1,1) \in \mathbb{Z}_{18} \oplus \mathbb{Z}_2$ has order 18; the largest order of any element in $\mathbb{Z}_6 \oplus \mathbb{Z}_6$ is $\text{lcm}(6,6) = 6$.

③ Let $H \leq G$ and $a, b \in G$.

ⓐ $aH = bH \iff a \in bH$

Proof: As $a = ae \in aH$, if $aH = bH$ then $a \in aH = bH$.
If $a \in bH$, then $a = bh$ for some $h \in H$. Thus

$$aH = (bh)H = b(hH) = bH. \quad \blacksquare$$

ⓑ $aH = bH$ or $aH \cap bH = \phi$

Proof: Suppose $aH \cap bH \neq \phi$ and let $c \in aH \cap bH$. Then by part ⓐ, $aH = cH = bH$. $\blacksquare$

④ Let G be a group of order 100 and $H \leq G$ with $|H| = 25$.

ⓐ If $a \in G$ has $|a| = 5$, then $H \cap \langle a \rangle = \langle a \rangle$ or $H \cap \langle a \rangle = \{e\}$.

Proof: Suppose $H \cap \langle a \rangle \neq \{e\}$. Then $a^i \in H$ for some $i = 1, 2, 3, 4$.
As 5 is prime, any such element a^i is a generator for $\langle a \rangle$. Hence:

$$\langle a \rangle = \langle a^i \rangle \leq H \cap \langle a \rangle \leq \langle a \rangle$$

and so $H \cap \langle a \rangle = \langle a \rangle$. $\blacksquare$

ⓑ Every element of order 5 in G is in H.

Proof: Suppose $a \in G$, $|a| = 5$ and $a \notin H$. Then by ⓐ $H \cap \langle a \rangle = \{e\}$.
Therefore

$$|\langle a \rangle H| = \frac{|\langle a \rangle| \cdot |H|}{|\langle a \rangle \cap H|} = \frac{5 \cdot 25}{1} = 125$$

but as $\langle a \rangle H \subseteq G$, and $|G| = 100$, this is a contradiction. $\blacksquare$

⑤ Let $S = \{\underbrace{(1,0)}_{a}, \underbrace{(0,1)}_{b}\} \subseteq \mathbb{Z}_3 \oplus \mathbb{Z}_4$.

$\longrightarrow$ a (1,0)

$\longrightarrow$ b (0,1)

$\mathrm{Cay}(\mathbb{Z}_3 \oplus \mathbb{Z}_4, S)$: