Exam 2 Key:

1) Consider the group $\mathbb{Z}_{20}$:

a) List all subgroups:

$\langle 0 \rangle, \langle 10 \rangle, \langle 5 \rangle, \langle 4 \rangle, \langle 2 \rangle, \langle 1 \rangle$

b) For each subgroup, list every generator:

$\langle 0 \rangle$ generator: 0

$\langle 10 \rangle$ generator: 10

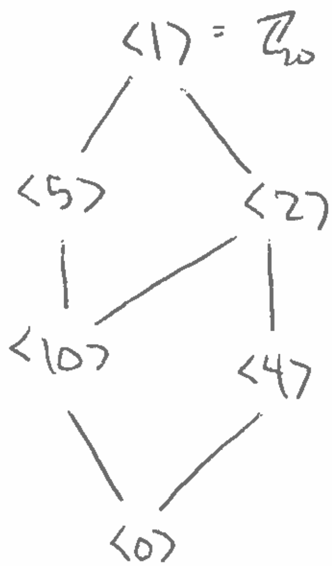
$\langle 5 \rangle$ generator: 5, 15

$\langle 4 \rangle$ generators: 4, 8, 12, 16

$\langle 2 \rangle$ generators: 2, 6, 14, 18

$\langle 1 \rangle$ generators: 1, 3, 7, 9, 11, 13, 17, 19

c) Determine the subgroup lattice:



2.) Consider the permutations

$$\alpha = (1\ 3\ 2\ 5\ 6)$$
$$\beta = (4\ 6\ 5\ 1\ 2)$$

in S_6.

a) $\beta^{-1} = (2\ 1\ 5\ 6\ 4)$

b) $\alpha\beta^{-1} = (1\ 3\ 2\ 5\ 6)(2\ 1\ 5\ 6\ 4)$

$$= (1\ 6\ 4\ 5)(2\ 3)$$

c) $\alpha\beta^{-1} = (16)(64)(45)(23)$

or

$(15)(14)(16)\ (23)$ or otherways too.

d) yes, $\alpha\beta^{-1} \in A_6$

e) $|\alpha\beta^{-1}| = \text{lcm}(4,2) = 4$.

3.) Explicitly show that $f: \mathbb{Z}_{15} \to \mathbb{Z}_{15}$ given by $f(x) = 7x \bmod 15$ is an isomorphism.

- f is a bijection as $g: \mathbb{Z}_{15} \to \mathbb{Z}_{15}$ given by $g(x) = 13x \bmod 15$ is an inverse function since $7 \cdot 13 = 91 \equiv 1 \bmod 15$.

- if $a, b \in \mathbb{Z}_{15}$ then

$$f(a+b) = 7(a+b) \bmod 15$$
$$= 7a + 7b \bmod 15$$
$$= f(a) + f(b)$$

therefore, f preserves the operation.

$\Rightarrow$ f is an automorphism

4.) Let G be a group, $g \in G$ and $z \in Z(G)$. Show $\phi_g = \phi_{zg}$.

Given any $x \in G$ we have:

$$\phi_{zg}(x) = (zg)x(zg)^{-1}$$

$$= zgxg^{-1}z^{-1}$$) as $z \in Z(G)$

$$= zz^{-1}gxg^{-1}$$

$$= gxg^{-1}$$

$$= \phi_g(x).$$

Thus $\phi_{zg} = \phi_g$. ∎

5.) Let G be a group and $a \in G$. If $|a^5| = 12$ determine the possibilities for $|a|$.

Let $|a| = n$. As $12 = |a^5| = n/\gcd(5,n)$ we have

$$n = 12 \gcd(5,n)$$

Since $\gcd(5,n)$ is either 1 or 5 we have that n is either 12 or 60.