WHITEHEAD MOVES FOR G–TREES

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Abstract. We generalize the familiar notion of a Whitehead move from Culler and Vogtmann's Outer space to the setting of deformation spaces of G–trees. Specifically, we show that there are two moves, each of which transforms a reduced G–tree into another reduced G–tree, that suffice to relate any two reduced trees in the same deformation space. These two moves further factor into three moves between reduced trees that have simple descriptions in terms of graphs of groups. This result has several applications.

1. Introduction

Whitehead automorphisms of F_n (the free group of rank n) generate the automorphism group $\mathrm{Aut}(F_n)$. These automorphisms were used by J.H.C. Whitehead to construct an algorithm to decide whether two given elements of F_n are related by an automorphism [17].

These automorphisms can be interpreted as certain moves on an n–rose whose fundamental group is marked with an isomorphism to F_n [12]. As such, they can be used to provide a path of marked n–roses connecting any two marked n–roses in Culler and Vogtmann's Outer space [4]. This is the space of marked metric graphs modulo homothety. By passing to the universal covers of the marked metric graphs, an alternative description of Outer space is the space of free minimal actions of F_n on metric simplicial trees, again modulo homothety.

Deformation spaces of G–trees (see [6]) are a generalization of Outer space, where the actions of a group G on a simplicial tree are allowed to have nontrivial stabilizers, but the set of elliptic subgroups (subgroups that fix points) is uniform throughout the deformation space. See [1, 2, 8, 11, 14, 15] for examples and applications of deformation spaces.

In a deformation space, the analogue of a rose in Outer space is a *reduced* tree (defined in Section 2). The purpose of this note is to find a finite set of moves, analogous to Whitehead moves, that will provide a path of reduced G–trees through a deformation space, connecting any two given reduced G–trees. This is achieved in Theorem 3.2, where it is shown that two particular moves suffice. These two moves are then decomposed into three simpler moves called *slide*, *induction*, and $\mathscr{A}^{\pm 1}$*–moves*. Our main theorem is the following.

[1]Partially supported by NSF grant DMS-0605137.

Theorem 1.1. *In a deformation space of G–trees, any two reduced trees are related by a finite sequence of slides, inductions, and $\mathscr{A}^{\pm 1}$–moves, with all intermediate trees reduced.*

In this paper, sliding a collection of edges over one edge is considered a single slide move. Of course, in the case of cocompact G–trees, the theorem is still valid using the traditional definition (sliding one edge orbit at a time); see Definition 2.2.

One immediate consequence of Theorem 1.1 is a strengthened form of the uniqueness theorem for JSJ decompositions of finitely generated groups [7]. Here, JSJ decompositions are meant in the sense of Rips and Sela [16], Dunwoody and Sageev [5], or Fujiwara and Papasoglu [9].

Corollary 1.2. *Any two JSJ decompositions of a finitely generated group are related by a finite sequence of slides, inductions, and $\mathscr{A}^{\pm 1}$–moves between reduced decompositions.*

At the end of the paper we discuss two further applications. One observation is that if the deformation space is *non-ascending* (see below) then induction and $\mathscr{A}^{\pm 1}$–moves cannot occur. Thus any two reduced trees are related by slide moves. This result has previously appeared as [8, Theorem 7.4] and [11, Theorem 7.2], and indeed our proof of Theorem 1.1 is similar in spirit to the proof given in [11]. The theorem also directly implies the rigidity theorem for G–trees [6, 10], in its most general form due to Levitt [13].

Lastly, Theorem 1.1 plays a significant role in the solution to the isomorphism problem for certain generalized Baumslag–Solitar groups. This work appears in [3].

2. Deformation spaces

A graph Γ is given by $(V(\Gamma), E(\Gamma), o, t, ^{-})$ where $V(\Gamma)$ are the vertices, $E(\Gamma)$ are the oriented edges, $o, t\colon E(\Gamma) \to V(\Gamma)$ are the originating and terminal vertex maps and $^{-}\colon E(\Gamma) \to E(\Gamma)$ is a fixed point free involution, which reverses the orientations of edges. An *edge path* $\gamma = (e_0, \ldots, e_k)$ is a sequence of edges such that $t(e_i) = o(e_{i+1})$ for $i = 0, \ldots, k-1$. A *loop* is an edge $e \in E(\Gamma)$ such that $o(e) = t(e)$. A *geometric edge* is a pair of the form $\{e, \bar{e}\}$. When we say that e, f are "distinct geometric edges" we mean that none of the oriented edges $e, \bar{e}, f, \bar{f}$ coincide.

Let G be a group. A *G–tree* is a simplicial tree T together with an action of G by simplicial automorphisms, without inversions (that is, $ge \neq \bar{e}$ for all $g \in G, e \in E(T)$). Two G–trees are considered equivalent if there is a G–equivariant isomorphism between them. The quotient graph T/G has the structure of a graph of groups with a marking (an identification of G with the fundamental group of the graph of groups). We call such graphs *marked graphs of groups*, or *marked graphs* for short.

Given a G–tree T, a subgroup $H \subseteq G$ is *elliptic* if it fixes a point of T. There are two moves one can perform on a G–tree without changing the elliptic subgroups, called *collapse and expansion moves*; they correspond to the natural isomorphism $A *_B B \cong A$. The exact definition is as follows.

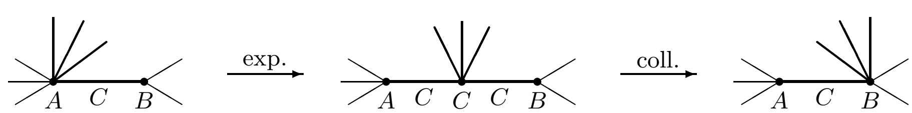
Definition 2.1. An edge e in a G–tree T is *collapsible* if $G_e = G_{o(e)}$ and its endpoints are not in the same orbit. If one collapses $\{e, \bar{e}\}$ and all of its translates to vertices, the resulting G–tree is said to be obtained from T by a *collapse move*. The reverse of this move is called an *expansion move*.

If Γ is the corresponding marked graph of groups, $e \in E(\Gamma)$ is collapsible if it is not a loop and the inclusion map $G_e \hookrightarrow G_{o(e)}$ is surjective. The marked graph obtained from Γ by collapsing e is denoted Γ_e. If $F \subset \Gamma$ is a forest whose edges can be collapsed iteratively, we denote the resulting marked graph Γ_F. A *non-trivial* forest is a forest containing at least one edge.

A G–tree (or marked graph) is *reduced* if it does not admit a collapse move. An *elementary deformation* is a finite sequence of collapse and expansion moves. Given a G–tree T, the *deformation space* $\mathscr{D}$ of T is the set of all G–trees related to T by an elementary deformation. If T is cocompact then $\mathscr{D}$ is equivalently the set of all G–trees having the same elliptic subgroups as T [6].

There are three special deformations that will be considered as basic moves. We define them below in terms of graphs of groups, but first we need some terminology. Suppose a graph of groups has an edge e which is a loop. Let A be the vertex group and B the edge group, with inclusion maps $i_0, i_1 \colon B \hookrightarrow A$. If one of these maps, say i_0, is an isomorphism, then e is an *ascending loop*. The *monodromy* is the composition $i_0^{-1} \circ i_1 \colon A \hookrightarrow A$. If the monodromy is not surjective then e is a *strict ascending loop*. A deformation space $\mathscr{D}$ is *ascending* if it contains a G–tree whose quotient graph of groups has a strict ascending loop. Otherwise it is called *non-ascending*.

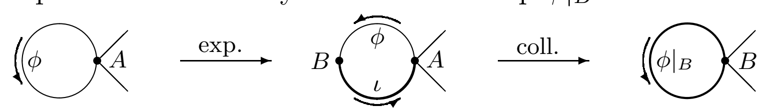
Definition 2.2. The deformation shown below is called a *slide move*. The edge groups of the edges that slide do not change. However, in order to perform the move, these edge groups must be contained in C (considered as subgroups of A before the move).



The set of edges that slide may have any cardinality. If this cardinality is finite, however, then the edges may of course be slid one at a time. Notice that in this situation, if the initial and final marked graphs are reduced, then so are the intermediate graphs when edges are slid separately.

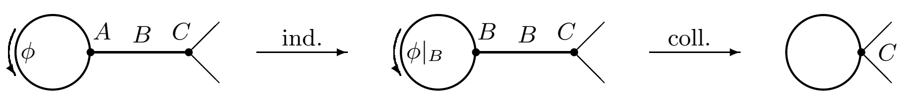
The edge carrying C is allowed to be a loop. In this case the only change to the graph of groups is in the inclusion maps of the edge groups to A. If a marking is given, however, then it will also change.

Definition 2.3. An *induction move* is an expansion and collapse along an ascending loop. In the diagram below the ascending loop has vertex group A and monodromy $\phi\colon A \hookrightarrow A$, and B is a subgroup such that $\phi(A) \subseteq B \subseteq A$. The map $\iota\colon B \hookrightarrow A$ is inclusion. The lower edge is expanded and the upper edge is collapsed, resulting in an ascending loop with monodromy the induced map $\phi|_B\colon B \hookrightarrow B$.



The reverse of this move is also considered an induction move. Notice that the vertex group may change, in contrast with slide moves.

Definition 2.4. An $\mathscr{A}^{-1}$*–move* is an induction followed by a collapse as shown below. The move is always non-trivial, and it has some requirements: the loop is an ascending loop with monodromy ϕ, with $\phi(A) \subseteq B \subseteq A$ (so that the induction can be performed), B is a proper subgroup of both A and C, and there are no other edges incident to the loop.



Note that before the move, the loop is a strict ascending loop, and after, the loop is not ascending. Thus an $\mathscr{A}^{-1}$–move removes an ascending loop, and its reverse, called an $\mathscr{A}$*–move*, adds one.

Remark 2.5. $\mathscr{A}^{\pm 1}$–moves preserve the property of being reduced. The same is not always true of slide or induction moves, unless one is in a non-ascending deformation space. Also, an $\mathscr{A}^{\pm 1}$–move can only occur in an ascending deformation space.

An n–rose in Outer space may be regarded as a marked graph of trivial groups with a single vertex. From this point of view, a Whitehead move is an expansion (as defined in 2.1) followed by a collapse of an edge other than the expanded one. The next definition generalizes this move. Let $\mathscr{D}$ be a deformation space of a group G.

Definition 2.6. Let $\Gamma, \Gamma' \in \mathscr{D}$ be reduced marked graphs. We say Γ and Γ' are *related by a type I Whitehead move* if there is a marked graph $\Gamma'' \in \mathscr{D}$ such that $\Gamma = \Gamma''_e$ and $\Gamma' = \Gamma''_{e'}$ for distinct geometric edges $e, e' \in E(\Gamma'')$. We say Γ and Γ' are *related by a type II Whitehead move* if there is a marked graph $\Gamma'' \in \mathscr{D}$ such that $\Gamma = \Gamma''_e$ and $\Gamma' = \Gamma''_{e' \cup f'}$ for distinct geometric edges $e, e', f' \in E(\Gamma'')$. Note that Whitehead moves are only defined between reduced marked graphs.

3. Finding and factoring Whitehead moves

Recall that Whitehead moves suffice to connect any two reduced marked graphs (roses) in Outer space [4]. We generalize this result to the setting of deformation spaces in Theorem 3.2 below. Then Theorem 1.1 is proved by expressing Whitehead moves in terms of slides, inductions, and $\mathscr{A}^{\pm 1}$–moves (Propositions 3.3 and 3.4).

We need to introduce some notation. If $\Gamma, \Gamma'' \in \mathscr{D}$ and $\Gamma = \Gamma''_F$ for some forest $F \subset \Gamma''$ and F_0 is a subforest of F, then we denote the marked graph Γ''_{F-F_0} by Γ^{F_0}. A forest $F_1 \subset \Gamma^{F_0}$ is collapsible if and only if $(\Gamma^{F_0})_{F_1} \in \mathscr{D}$, and when this occurs, we will abbreviate $(\Gamma^{F_0})_{F_1}$ as $\Gamma^{F_0}_{F_1}$. Given a graph of groups Γ and an edge $e \in E(\Gamma)$, the inclusion map $G_e \hookrightarrow G_{o(e)}$ may or may not be surjective. We assign a *label* to e, which is "$=$" if $G_e \hookrightarrow G_{o(e)}$ is surjective and "$\neq$" otherwise.

Lemma 3.1. *Let $\Gamma, \Gamma' \in \mathscr{D}$ be reduced marked graphs and suppose that there is a marked graph $\Gamma'' \in \mathscr{D}$ such that $\Gamma = \Gamma''_F$ and $\Gamma' = \Gamma''_{F'}$ for non-trivial finite forests $F, F' \subset \Gamma''$ that do not share an edge. Then there are edges $e \in F, e' \in F'$ such that one of the following holds:*

(a) $\Gamma^e_{e'} \in \mathscr{D}$ *and* $\Gamma^e_{e'}$ *is reduced,*
(b) *there is an edge $f' \in \Gamma''$ such that $\Gamma^e_{e' \cup f'} \in \mathscr{D}$ and $\Gamma^e_{e' \cup f'}$ is reduced,*
(c) $\Gamma'^{e'}_e \in \mathscr{D}$ *and* $\Gamma'^{e'}_e$ *is reduced, or*
(d) *there is an edge $f \in \Gamma''$ such that $\Gamma'^{e'}_{e \cup f} \in \mathscr{D}$ and $\Gamma'^{e'}_{e \cup f}$ is reduced.*

Note that the lemma is symmetric in Γ and Γ'.

Proof. We begin with the following claim.

Claim. If there are edges $e \in F$, $e' \in F'$ such that e' is collapsible in Γ^e, then conclusion (a) or (b) holds.

Proof of Claim. Replacing e by $\bar{e}$ if necessary, we may assume that e has label $=$ in Γ^e. Then, since $\Gamma = \Gamma^e_e$ is reduced, every collapsible edge in Γ^e must be incident to $o(e)$. Furthermore, every such edge f with $o(f) = o(e)$ has one of three types:

type 1: $t(f) \neq t(e)$ (which implies that $\bar{f}$ has label $\neq$)
type 2: $t(f) = t(e)$ and $\bar{f}$ has label $\neq$
type 3: $t(f) = t(e)$ and $\bar{f}$ has label $=$.

Note that collapsing a type 2 edge always results in a reduced marked graph. Also, after collapsing a type 1 edge, type 3 edges remain collapsible and the other types become non-collapsible. Similarly, after collapsing a type 3 edge, type 1 edges remain collapsible and the others become non-collapsible.

If e' is of type 2 then conclusion (a) holds. In fact, by the observations above, the only way $\Gamma^e_{e'}$ can fail to be reduced is if Γ^e has collapsible edges f_1 of type 1 and f_3 of type 3, one of which is e'. Then f_3 is collapsible in $\Gamma^e_{f_1}$, implying that $\Gamma^e_{f_1 \cup f_3} \in \mathscr{D}$; and $\Gamma^e_{f_1 \cup f_3}$ is reduced, establishing (b). □

Returning to the lemma, we proceed by considering various configurations of the forests F and F'. Since $F \subseteq \Gamma''$ is collapsible, in each component F_0 of F there is a maximal subtree $F_1 \subseteq F_0$ such that every edge in F_1 has label $=$ and every edge

$e \in F_0 - F_1$ which separates $o(e)$ from F_1 also has label $=$. We call F_1 the *maximal stable subtree* of F_0.

Let e' be any edge of F' with label $=$, and suppose that e' does not map to a loop in Γ. Then there is a component F_0 of F containing $o(e')$ but not $t(e')$. Let F_1 be the maximal stable subtree of F_0. Notice that $o(e') \notin F_1$, since otherwise e' would be collapsible in Γ. Let e be the first edge in the path in F_0 from $o(e')$ to F_1. Note that collapsing $F - \{e\}$ does not enlarge the vertex group $G_{o(e')}$, and so e' is collapsible in Γ^e. By the Claim, conclusion (a) or (b) holds. Similarly, by symmetry, if there is an edge in F with label $=$ which does not map to a loop in Γ', then conclusion (c) or (d) holds.

Therefore, we may assume that every edge of F' maps to a loop in Γ, and every edge of F maps to a loop in Γ'. Now let F_0 be a component of F and $F_1 \subseteq F_0$ its maximal stable subtree. Choose a vertex $v \in F_1$. There is an edge in F_0 incident to v, and there is a path in F' joining the endpoints of this edge (since it maps to a loop in Γ'). In particular, there is an edge $e' \in F'$ with initial vertex v. Now let γ be the path in F_0 from v to $t(e')$, which exists since e' maps to a loop in Γ. Let e be the final edge of γ, with $t(e) = t(e')$. Because $v \in F_1$, collapsing $F - \{e\}$ does not enlarge the vertex groups at $v = o(e')$ or at $t(e')$. Hence e' is collapsible in Γ^e, and by the Claim we are finished. $\square$

Theorem 3.2. *Any two reduced marked graphs in $\mathscr{D}$ are related by a sequence of Whitehead moves.*

Proof. Let $\Gamma, \Gamma' \in \mathscr{D}$ be reduced marked graphs. First we consider a special case, when Γ and Γ' satisfy the hypotheses of Lemma 3.1. In this case the theorem is proved by induction on the number of edges in $F \cup F'$, as follows.

Apply Lemma 3.1 and suppose that conclusion (a) holds. Then Γ and $\Gamma^e_{e'}$ are related by a Whitehead move. Also, $\Gamma''_{e'}$ collapses to $\Gamma^e_{e'}$ and to Γ', by collapsing the forests $F - \{e\}$ and $F' - \{e'\}$ respectively. If both of these forests have no edges then $\Gamma^e_{e'} = \Gamma'$ and we are done. Otherwise, since $\Gamma^e_{e'}$ and Γ' are reduced, both forests are non-trivial. Hence $\Gamma^e_{e'}$ and Γ' satisfy the hypotheses of Lemma 3.1, and by induction, they are related by Whitehead moves. The cases (b), (c), (d) are similar.

For the general case, start with an elementary deformation from Γ to Γ'. By cancelling any trivial expansion-collapse pairs, and inserting trivial collapse-expansion pairs, the deformation can be put into the form

$$\Gamma = \Gamma_0 \leftarrow \Gamma_1 \rightarrow \Gamma_2 \leftarrow \Gamma_3 \rightarrow \cdots \leftarrow \Gamma_{2n-1} \rightarrow \Gamma_{2n} = \Gamma'$$

where each Γ_{2i} is reduced, each arrow is a sequence of collapses, and the two forests being collapsed in $\Gamma_{2i} \leftarrow \Gamma_{2i+1} \rightarrow \Gamma_{2i+2}$ are finite and have no shared edges. Now Γ_{2i} and Γ_{2i+2} satisfy the hypotheses of Lemma 3.1 for each i, and are related by Whitehead moves, by the special case. $\square$

Proposition 3.3. *Any type I Whitehead move is a composition of slides and inductions, where the intermediate marked graphs are reduced. Moreover, the only edge being slid over is one of the edges which is collapsed in the Whitehead move.*

Proof. Let $\Gamma, \Gamma' \in \mathscr{D}$ be reduced marked graphs such that $\Gamma = \Gamma''_e$ and $\Gamma' = \Gamma''_{e'}$. Orient e, e' so that $o(e) = o(e')$. Let $\{f_\alpha\}$ be the other edges with initial vertex $o(e)$.

If $e \cup e'$ is not a cycle in Γ'' then e and e' both have label $=$. Then Γ' is obtained from Γ by sliding the collection of edges $\{f_\alpha\}$ over e'. There is no intermediate marked graph in this case.

Now suppose that $e \cup e'$ is a cycle and e has label $=$ in Γ''. If e' also has label $=$, then as above, Γ' is obtained from Γ by sliding the collection of edges $\{f_\alpha\}$ over the loop e'. Otherwise we can assume that $\bar{e}, e'$ have label $\neq$ and $\bar{e}'$ has label $=$. Then Γ' is obtained from Γ by an induction move (in which the same edge e is expanded, but the edges f_α are left at $t(e')$), followed by a slide of the collection of edges $\{f_\alpha\}$ over the loop $\bar{e}$. Since Γ' is reduced, each f_α which is not a loop has label $\neq$ in Γ''. This implies that the intermediate marked graph is reduced. □

Proposition 3.4. *Any type II Whitehead move is a composition of slides and $\mathscr{A}$– or $\mathscr{A}^{-1}$–moves, where the intermediate marked graphs are reduced.*

Proof. Let $\Gamma, \Gamma' \in \mathscr{D}$ be reduced marked graphs such that $\Gamma = \Gamma''_e$ and $\Gamma' = \Gamma''_{e' \cup f'}$. This is the situation of conclusion (b) of Lemma 3.1. The proof of the Claim shows that e' and f' are of types 1 and 3. After renaming, these edges of Γ'' must have the configuration shown in Figure 1. The labels are as shown because Γ and Γ' are reduced. Let $\{g_\alpha\}$ be the other edges with initial vertex $t(e)$, and $\{h_\beta\}$ the other

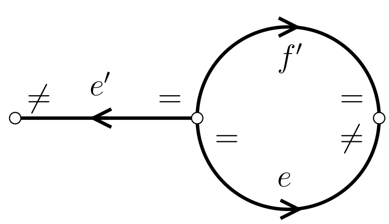


FIGURE 1.

edges with initial vertex $o(e)$. Now Γ' is obtained from Γ by first sliding the collection $\{h_\beta\}$ over $(\bar{f}', e')$, then sliding $\{g_\alpha\}$ over e', and then performing an $\mathscr{A}^{-1}$–move. It is easy to verify that the intermediate marked graphs are reduced. □

Theorem 1.1 now follows directly from Theorem 3.2 and Propositions 3.3 and 3.4.

The next result follows easily from Theorem 1.1, as explained in the introduction. The second statement is included for use in [3].

Corollary 3.5. *In a non-ascending deformation space of G–trees, any two reduced trees are related by a finite sequence of slide moves, with all intermediate trees reduced. Moreover, if e is an edge of T and a deformation from T to T' never collapses e, then there is a sequence of slide moves from T to T' in which no edge slides over e.*

Proof. Start with a deformation between reduced G–trees T, T' in $\mathscr{D}$. There is a sequence of Whitehead moves joining T to T' by Theorem 3.2, and these moves are all of type I since $\mathscr{D}$ is non-ascending (a type II Whitehead move cannot occur by Proposition 3.4). Recall that in the proof of Theorem 3.2, each time a type I Whitehead move is factored out, the expansion and collapse comprising that move were already present in the original elementary deformation. Thus, by Proposition 3.3, there is a sequence of slide and induction moves from T to T', and the only edges that are slid over were expanded or collapsed in the original deformation. Lastly, there are no induction moves since $\mathscr{D}$ is non-ascending. □

The rigidity theorem for G–trees [6, 10, 13] also follows quickly from Theorem 1.1. Recall that a G–tree $T \in \mathscr{D}$ is *rigid* if it is the only reduced G–tree in $\mathscr{D}$.

Corollary 3.6 (Levitt)**.** *A G–tree that is not the Bass–Serre tree of an ascending HNN-extension is rigid if and only if, for any two edges e, f such that $o(e) = o(f) = v$ and $G_e \subseteq G_f$, one of the following conditions holds:*

(a) $e \in Gf$,

(b) $e \in G\bar{f}$ *and* $G_e = G_f$, *or*

(c) *there is an edge f' such that $o(f') = v$, $f' \in G\bar{f}$ and $G_f = G_{f'} = G_v$ and there are only three G_v–orbits of edges at v.*

Proof. By Theorem 1.1 it is clear that such a G–tree is rigid if and only if it does not admit a slide, induction or $\mathscr{A}^{\pm 1}$–move resulting in a different G–tree. Given e, f as above, if $e \notin Gf \cup G\bar{f}$ then there is a (possibly trivial) slide move of e over f. This slide move is trivial only under the conditions of (c). If $e \in G\bar{f}$ and $G_e \neq G_f$ then the G–tree admits an $\mathscr{A}^{\pm 1}$–move or induction move. Note that an induction move is always non-trivial, except possibly in the case of an ascending HNN extension. Thus, if T is rigid, then e and f satisfy one of the three conditions. For the converse, if T admits a slide, induction or $\mathscr{A}^{\pm 1}$–move, then there is a pair of edges e, f that do not satisfy any of the three conditions. □

Theorem 1.1 implies that a G–tree that is the Bass–Serre tree of an ascending HNN-extension is rigid if and only if it does not admit a nontrivial induction move. See [13, Theorem 2] for algebraic conditions on the monodromy $\phi\colon A \hookrightarrow A$ characterizing when the G–tree does not admit a nontrivial induction move.

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