TWISTING OUT FULLY IRREDUCIBLE AUTOMORPHISMS

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Abstract. By a theorem of Thurston, in the subgroup of the mapping class group generated by Dehn twists around two curves which fill, every element not conjugate to a power of one of the twists is pseudo-Anosov. We prove an analogue of this theorem for the outer automorphism group of a free group.

1. Introduction

A *fully irreducible* element of the outer automorphism group $\mathrm{Out}\, F_k$ of a free group F_k is characterized by the property that no nontrivial power fixes the conjugacy class of a proper free factor of F_k. Considered to be analogous to pseudo-Anosov elements of the mapping class group (see [7] or [13]), fully irreducible elements play a similarly important role in the study of $\mathrm{Out}\, F_k$. Levitt and Lustig [26] showed for instance that fully irreducible elements exhibit North-South dynamics on the closure of Culler–Vogtmann's Outer space, the projectivized space of minimal very small actions of F_k on $\mathbb{R}$–trees [4, 9]. More recently Algom-Kfir [1] proved that axes of fully irreducibles in Outer space, equipped with the Lipschitz metric, are strongly contracting, indicating that this class of outer automorphisms should be useful towards understanding the geometry of $\mathrm{Out}\, F_k$.

In this paper we present a method for constructing fully irreducible elements of $\mathrm{Out}\, F_k$. Our approach is to replicate the following result of Thurston concerning pseudo-Anosov mapping classes: a pair of Dehn twists around filling simple closed curves generate a nonabelian free group in which any element not conjugate to a power of one of the twists is pseudo-Anosov [40]. The irreducible outer automorphisms we construct have the additional property of being *atoroidal*; that is, none of their nontrivial powers fix a conjugacy class of F_k. By theorems of Bestvina–Feighn [5], Brinkmann [6], and Gersten [14], the atoroidal elements of $\mathrm{Out}\, F_k$ are precisely the *hyperbolic* elements, consisting of exactly those elements with hyperbolic mapping tori, and so we will use only the latter term.

Before stating precisely our main theorem, we briefly recall some known constructions of fully irreducible elements of $\mathrm{Out}\, F_k$.

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Geometric: By Thurston's theorem, we obtain pseudo-Anosov homeomorphisms from two Dehn twists around filling curves on a surface S with a single boundary component. From an identification $\pi_1(S) \cong F_k$, any such pseudo-Anosov induces a fully irreducible outer automorphism of F_k. Such an outer automorphism is necessarily not hyperbolic as the conjugacy class of the element of F_k corresponding to the boundary component of S is periodic.

Homological: As in the case of the mapping class group [7, 29], there is a homological criterion that ensures an outer automorphism is fully irreducible. Namely, Gersten and Stallings [15] gave algebraic criteria for fully irreducibility, providing sufficient conditions in terms of the matrix corresponding to the action of the outer automorphism on the homology of F_k. This provides examples of hyperbolic fully irreducible elements, but the action on homology is necessarily nontrivial.

Our construction begins with an analogy to surfaces: a simple closed curve on a surface determines a splitting of the surface group over the cyclic subgroup generated by the curve. For $\mathrm{Out}\, F_k$, the role of a simple closed curve can be taken by a splitting of F_k over a cyclic subgroup. We prove using an appropriate notion of Dehn twist automorphism (defined by a splitting of F_k over a cyclic subgroup) and of filling splittings:

Theorem 5.3. *Let δ_1 and δ_2 be the Dehn twists of F_k for two filling cyclic splittings of F_k. Then there exists $N = N(\delta_1, \delta_2)$ such that for $m, n > N$:*

(1) *$\langle \delta_1^m, \delta_2^n \rangle$ is isomorphic to the free group on two generators; and*
(2) *if $\phi \in \langle \delta_1^m, \delta_2^n \rangle$ is not conjugate to a power of either δ_1^m or δ_2^n, then ϕ is a hyperbolic fully irreducible element of $\mathrm{Out}\, F_k$.*

Kapovich-Lustig [23] and Hamenstädt [18] have recently, using different methods, given another construction of hyperbolic fully irreducible elements in $\mathrm{Out}\, F_k$. Namely, they show that given two hyperbolic fully irreducible elements $\phi, \psi \in \mathrm{Out}\, F_k$, then either the subgroup $\langle \phi, \psi \rangle$ is virtually cyclic, or there is a constant $N = N(\phi, \psi)$ such that for all $m, n \geq N$ the subgroup $\langle \phi^m, \psi^n \rangle$ is isomorphic to the free group on two generators and every nontrivial element of the subgroup is hyperbolic and fully irreducible.

Theorem 5.3 produces new examples of fully irreducible elements, not attained by previous methods. For instance, we can construct examples of hyperbolic (and therefore obtainable by Thurston's theorem) fully irreducible elements that act trivially on homology. Papadopoulos used Thurston's construction of pseudo-Anosov homeomorphisms that act trivially on homology to construct for any symplectic matrix in $\mathrm{Sp}(2g, \mathbb{Z})$ a pseudo-Anosov homeomorphism whose action on the first homology of the surface is the given matrix [31]. In [8], we use Theorem 5.3 and the techniques developed within this current paper, to construct for any matrix in $\mathrm{GL}(k, \mathbb{Z})$ a fully irreducible hyperbolic element whose action on the first homology of F_k is the given matrix.

Consider the descending sequence of subgroups of $\operatorname{Aut} F_k$ given by

$$\operatorname{Aut} F_k \to \operatorname{Aut}(F_k/\Gamma^{i+1}(F_k))$$

where $\Gamma^2(F_k) = [F_k, F_k]$, the commutator subgroup of F_k, and $\Gamma^{i+1}(F_k) = [F_k, \Gamma^i(F_k)]$. The *Johnson filtration* of $\operatorname{Out} F_k$ is the induced sequence $J_k^1 \supset J_k^2 \supset \cdots$ of subgroups of $\operatorname{Out} F_k$; the group J_k^1 is analogous to the *Torelli subgroup* of the mapping class group of a surface. Observe that $[J_k^i, J_k^i] \subset J_k^{i+1}$, so that by applying Theorem 5.3 we have:

Corollary 1.1. *For $k \geq 3$, there exist hyperbolic fully irreducible elements arbitrarily deep in the Johnson filtration for* $\operatorname{Out} F_k$.

To prove Theorem 5.3, we use methods necessarily very different from Thurston's, which employed much of the rich geometry of Teichmüller space. Our argument is based closely on an alternate, more combinatorial proof due to Hamidi-Tehrani [19] that applies a variant on the usual ping pong argument to the set of simple closed curves on a surface. Much of the work in our paper is concerned with setting up a suitable substitute for the intersection number of two simple closed curves on a surface, a key ingredient in Hamidi-Tehrani's argument.

Observe that the intersection number between two curves α and β on a surface S is equal to the combinatorial translation length of the element $\alpha \in \pi_1(S)$ on the dual tree to lifts of β in the hyperbolic plane $\mathbb{H}^2$. This dual tree is exactly the Bass–Serre tree for the splitting of the surface group over the cyclic subgroup generated by β. We formulate a generalization of intersection numbers to finitely generated subgroups H of F_k. In the following, T^H denotes a minimal non-empty H–invariant subtree of T.

Definition 2.2. *Suppose H is a finitely generated free group that acts on a simplicial tree T such that the stabilizer of an edge is either trivial or cyclic. The* free volume $\operatorname{vol}_T(H)$ *of H with respect to T is the number of edges of the graph of groups decomposition T^H/H with trivial stabilizer.*

It should be remarked that different notions of intersection number have been developed by Scott–Swarup [33], Guirardel [17], and Kapovich–Lustig [24], but that ours has been tailored to suit the needs of our theorem.

The main ingredient in our proof of Theorem 5.3 is the following result concerning the growth of the free volume under iterations of a Dehn twist:

Theorem 4.6. *Let δ_1 be a Dehn twist associated to the very small cyclic tree T_1 with edge stabilizers generated by conjugates of the element c_1 and let T_2 be any other very small cyclic tree. Then there exists a constant $C = C(T_1, T_2)$ such that for any finitely generated malnormal or cyclic subgroup $H \subseteq F_k$ with $\operatorname{rank}(H) \leq R$ and $n \geq 0$ the following hold:*

$$\operatorname{vol}_{T_2}(\delta_1^{\pm n}(H)) \geq \operatorname{vol}_{T_1}(H)\big(n\ell_{T_2}(c_1) - C\big) - M \operatorname{vol}_{T_2}(H)$$
$$\operatorname{vol}_{T_2}(\delta_1^{\pm n}(H)) \leq \operatorname{vol}_{T_1}(H)\big(n\ell_{T_2}(c_1) + C\big) + M \operatorname{vol}_{T_2}(H)$$

where M is the constant from Proposition 4.5.

The inequalities in Theorem 4.6 should be compared with the following inequality from [13] (see also [21]) for simple closed curves α, β, and γ on a surface:

$$|i(\delta_\beta^{\pm n}(\gamma), \alpha) - ni(\gamma, \beta)i(\alpha, \beta)| \leq i(\gamma, \alpha)$$

Here $i(\cdot, \cdot)$ denotes the geometric intersection number of two simple closed curves, and δ_β is the Dehn twist around the curve β. An asymptotic version of Theorem 4.6 for cyclic subgroups appears as a special case of Cohen and Lustig's "Skyscraper Lemma" [9, Lemma 4.1].

Although it is not essential to our main theorem, we describe a property of our notion of intersection number which likens it to intersection number for surfaces, as we consider it of independent interest. Recall that if α and β are simple closed curves that fill a surface S, and if σ is any hyperbolic metric on S, then there is constant K such that for any simple closed curve γ on S:

$$\frac{1}{K}\ell_\sigma(\gamma) \leq i(\alpha, \gamma) + i(\beta, \gamma) \leq K\ell_\sigma(\gamma) \tag{1.1}$$

where $\ell_\sigma(\gamma)$ is the length of the geodesic representing γ with respect to the metric σ.

Now recall that Culler–Vogtmann's Outer space CV_k consists of minimal discrete free actions of F_k on $\mathbb{R}$–trees, normalized such that the sum of the lengths of the edges in the quotient graph is 1 [12]. A point of CV_k, or its unprojectivized version cv_k, plays the role of a marked hyperbolic metric on S. There is a compactification $\overline{CV}_k$ [11] which is covered by $\overline{cv}_k$. The space $\overline{cv}_k$ is the space of minimal *very small* actions of F_k on $\mathbb{R}$–trees [4, 9]. Kapovich and Lustig showed that if T_1 and T_2 are trees in $\overline{cv}_k$ that are "sufficiently transverse", then for any tree $T \in cv_k$ there is a constant K such that for any element $g \in F_k$:

$$\frac{1}{K}\ell_T(g) \leq \ell_{T_1}(g) + \ell_{T_2}(g) \leq K\ell_T(g) \tag{1.2}$$

where $\ell_T(\cdot)$ is the translation length function for the tree T. We show a different generalization of (1.1).

Theorem 6.1. *Let T_1 and T_2 be two very small cyclic trees for F_k that fill and $T \in cv_k$. Then there is a constant K such that for any proper free factor or cyclic subgroup $X \subset F_k$:*

$$\frac{1}{K}\operatorname{vol}_T(X) \leq \operatorname{vol}_{T_1}(X) + \operatorname{vol}_{T_2}(X) \leq K\operatorname{vol}_T(X).$$

Our paper is organized as follows. Section 2 recalls well-known facts about $\operatorname{Out} F_k$ along with the definitions needed. The only new material in this section is a discussion on "filling" cyclic trees. In particular, we present a construction for producing filling cyclic trees when $k \geq 3$. In Section 3 we describe how to compute the free volume of a finitely generated subgroup of F_k with respect to a cyclic tree. This should be compared to the "no bigon" condition for computing intersection numbers between simple closed curves on a surface. The main result of Section 4 is to give a proof of Theorem 4.6. The Hamidi-Tehrani ping pong argument is applied in Section 5 to prove Theorem 5.3. Finally, in Section 6 we prove Theorem 6.1.

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2. Preliminaries

2.1. **Basics.** Let F_k denote the rank k non-abelian free group. For a basis $\mathcal{A} = \{x_1, \dots, x_k\}$ we fix a marked k–petaled rose $\Lambda = \Lambda_{\mathcal{A}}$: a graph with one vertex and k oriented petals with an identification to the set $\{x_1, \dots, x_k\}$, thus equipped with an isomorphism $F_k \to \pi_1(\Lambda, \text{vertex})$. A marking of a graph $\mathcal{G}$ with $\pi_1(\mathcal{G}) \cong F_k$ is a homotopy equivalence $\Lambda \to \mathcal{G}$. An outer automorphism ϕ of the free group determines a homotopy equivalence $\Phi\colon \Lambda \to \Lambda$. This gives a right action of $\operatorname{Out} F_k$ on the set of markings by precomposing the homotopy equivalence $\Lambda \to \mathcal{G}$ by Φ; that is, ϕ acts by changing the marking. The universal cover of a marked graph $\mathcal{G}$ is a tree $\tilde{\mathcal{G}}$ equipped with a free action of F_k; the set of such trees inherits the right action of $\operatorname{Out} F_k$, which coincides with the action of $\operatorname{Out} F_k$ on Outer space CV_k or cv_k.

Given a simplicial map $f_0\colon \mathcal{H}_0 \to \mathcal{G}$ between graphs, either it is an immersion (i.e., locally injective), or there is some pair of edges e_1, e_2 sharing a common initial vertex in $\mathcal{H}_0$ that have the same image under f_0. In case of the latter, let $\mathcal{H}_1$ be the quotient graph of $\mathcal{H}_0$ obtained by identifying e_1 with e_2; then f_0 descends to a well-defined map $f_1\colon \mathcal{H}_1 \to \mathcal{G}$. We say that the map $f_1 : \mathcal{H}_1 \to \mathcal{G}$ is obtained from $f_0 : \mathcal{H}_0 \to \mathcal{G}$ by a *fold*. Folding can be iterated until the resulting simplicial map $f\colon \mathcal{H} \to \mathcal{G}$ is an immersion of graphs [37]. In the case that $\mathcal{H}$ has valence one vertices, adjacent edges can be iteratively pruned from $\mathcal{H}$ to obtain a core graph $\mathcal{H}_{core}$ (a graph in which every edge belongs to at least one cycle) to which f restricts to a map $f_{core}\colon \mathcal{H}_{core} \to \mathcal{G}$.

Using folding, we can associate to the conjugacy class of a finitely generated subgroup H of F_k an immersion of a core graph $\mathcal{G}^H_{\mathcal{A}} \to \Lambda_{\mathcal{A}}$. Fix a basis for H, and let $\mathcal{H}$ be a rank(H)–petaled rose, where each petal is subdivided into edges labeled according to the associated word in the basis $\mathcal{A}$. The labels determine a map $\mathcal{H} \to \Lambda_{\mathcal{A}}$; after a series of folds, the resulting map is an immersion of graphs which can be pruned to obtain an immersion of the core graph $\mathcal{G}^H_{\mathcal{A}} \to \Lambda_{\mathcal{A}}$. The immersion $\mathcal{G}^H_{\mathcal{A}} \to \Lambda_{\mathcal{A}}$ does not depend on the initial graph $\mathcal{H}$. We refer to Stallings' paper [37] for more details.

For a basis $\mathcal{A}$ of F_k and element $x \in F_k$, we let $|x|_{\mathcal{A}}$ denote the reduced word length of x with respect to the basis $\mathcal{A}$. When dealing with word length in free groups the following lemma due to Cooper is indispensable:

Lemma 2.1 (Bounded cancellation [10])**.** *Suppose $\mathcal{A}_1$ and $\mathcal{A}_2$ are bases for the free group F_k. There is a constant $C = C(\mathcal{A}_1, \mathcal{A}_2)$ such that if w and w' are two elements of F_k where:*

$$|w|_{\mathcal{A}_1} + |w'|_{\mathcal{A}_1} = |ww'|_{\mathcal{A}_1}$$

then

$$|w|_{\mathcal{A}_2} + |w'|_{\mathcal{A}_2} - |ww'|_{\mathcal{A}_2} \leq 2C.$$

We denote by $BCC(\mathcal{A}_1, \mathcal{A}_2)$ the bounded cancellation constant; that is, the minimal constant C satisfying the lemma for $\mathcal{A}_1$ and $\mathcal{A}_2$. In other words, if ww' is a reduced word in $\mathcal{A}_1$, $w = \prod_{i=1}^{m} x_i$ and $w' = \prod_{i=1}^{m'} x'_i$ where $x_i, x'_i \in \mathcal{A}_2$, then for $C = BCC(\mathcal{A}_1, \mathcal{A}_2)$ the subwords $x_1 \cdots x_{m-C-1}$ and $x'_{C+1} \cdots x'_{m'}$ appear as subwords of ww' when considered as a word in $\mathcal{A}_2$.

Besides the free simplicial F_k–actions arising from marked graphs, we will also consider free group actions on simplicial trees that arise as Bass-Serre trees of splittings of F_k over cyclic subgroups. In general, for an F_k–tree T the action when restricted to a finitely generated subgroup H is not minimal, i.e., there is a non-empty proper H–invariant subtree. When H does not fix a point in T, we let T^H denote the smallest non-empty proper H–invariant subtree of T. Such a subtree is characterized as the union of the axes of all of the elements of H that do not fix a point in T [11]. When H fixes a subtree of T pointwise, we let T^H be any point of T fixed by H. We denote by $\ell_T(x)$ the translation length of the element $x \in F_k$ in the tree T.

2.2. **Dehn twist automorphisms.** The simplest type of homeomorphism of a surface is a *Dehn twist*. These homeomorphisms are supported on an annular neighborhood of a simple closed curve and are defined by cutting the surface open along the curve and regluing after twisting one side by 2π. Algebraically, a simple closed curve on a surface $\alpha \subset S$ determines a splitting of the fundamental group $\pi_1(S)$ either as an amalgamated free product $\pi_1(S_1) *_{\langle\alpha\rangle} \pi_1(S_2)$, if α is separating ($S - \alpha = S_1 \coprod S_2$); or as an HNN-extension $\pi_1(S')*_{\langle\alpha\rangle}$, if α is nonseparating ($S - \alpha = S'$).

By analogy, we now define a *Dehn twist automorphism*; see [9, 25, 32] for their use in various other settings. First consider the splitting of $F_k = A *_{\langle c\rangle} B$ which expresses F_k as an amalgamation of two free groups over a cyclic group. Define an automorphism δ of F_k by:

$$\forall a \in A \qquad \delta(a) = a$$
$$\forall b \in B \qquad \delta(b) = cbc^{-1}.$$

The automorphism δ acts trivially on homology and therefore belongs to the subgroup IA_k. Dehn twist automorphisms arising from amalgamations over $\mathbb{Z}$ should be considered analogous to a Dehn twist around a separating simple closed curve on a surface.

We similarly obtain an automorphism δ from an HNN-extension of the form

$$F_k = A*_{\mathbb{Z}} = \langle A, t \mid t^{-1}a_0t = a_1\rangle$$

for $a_0, a_1 \in A$ by:

$$\begin{aligned} \forall a \in A \qquad & \delta(a) = a \\ & \delta(t) = a_0t. \end{aligned}$$

Automorphisms arising from HNN-extensions should be compared to a Dehn twist around a nonseparating curve on a surface.

As there is no way to orient a free group, we cannot speak of "left Dehn twists" or "right Dehn twists" as for surfaces. Thus when we say "δ is a Dehn twist associated to the cyclic tree T," we are referring to one of the above defined automorphisms for the given edge group.

From Bass–Serre theory, a splitting of F_k over $\mathbb{Z}$ defines an action of F_k on a tree T, the *Bass–Serre tree* of the splitting (see [2] or [35]). We will refer to such F_k–trees as *cyclic*. In a certain sense, cyclic trees for F_k correspond to simple closed curves on a surface. In particular, Dehn twist automorphisms associated to cyclic trees generate an index two subgroup of $\mathrm{Aut}\, F_k$ (the subgroup which induces an action of $\mathrm{SL}_k(\mathbb{Z})$ on homology). Note that if δ is the Dehn twist automorphism associated to the cyclic tree T, then δ preserves the action of F_k on T, i.e., there is an isometry $h_\delta \colon T \to T$ such that $\forall g \in F_k$ and $\forall x \in T$ we have $h_\delta(gx) = \delta(g)h_\delta(x)$. In particular, $\ell_T(\delta(g)) = \ell_T(g)$ for all $g \in F_k$.

We are primarily interested in the *outer* automorphism group of F_k, and so in the sequel a Dehn twist will refer to an element of $\mathrm{Out}\, F_k$ which is induced by a Dehn twist automorphism in $\mathrm{Aut}\, F_k$.

2.3. **Guirardel's core and free volume.** Our strategy for proving Theorem 5.3 requires some notion of intersection number between a cyclic tree T and a free factor or cyclic subgroup $X \subset F_k$. To motivate our choice of intersection number we re-examine intersections of curves on surfaces.

For two simple closed curves $\alpha, \beta \subset S$, the intersection number $i(\alpha, \beta) = \ell_{T_\alpha}(\beta)$ where T_α is the Bass-Serre tree dual to the splitting of $\pi_1(S)$ over α. Hence our notion of intersection number between a cyclic tree T and a cyclic group $X = \langle g \rangle$ should be equal to $\ell_T(g)$. Given a subsurface $S_0 \subset S$ and a simple closed curve $\alpha \subset S$, there is an obvious way to define an intersection number $i(\alpha, S_0)$ by considering the boundary ∂S_0 and setting $i(\alpha, S_0) = i(\alpha, \partial S_0)$ (when ∂S_0 is not connected we take the sum over the individual components). This is exactly twice the number of arc components in $\alpha \cap S_0$.

Using the *Guirardel core*, one can associate a "subsurface" to a free factor relative to a pair of cyclic trees T_1 and T_2. As the Guirardel core is not used in later sections, we will not give the complete definition; for more details see

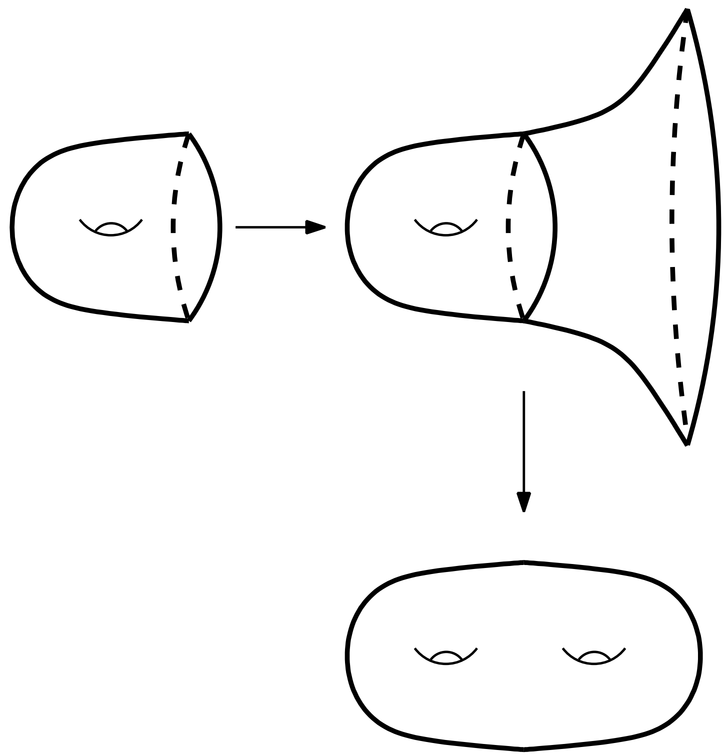

FIGURE 1. The map $\mathcal{C}^X/X \to \mathcal{C}/X \to \mathcal{C}/F_k$.

[17] or [3]. For our purposes we only need to know that the core $\mathcal{C} \subset T_1 \times T_2$ is an F_k–invariant subset (with respect to the diagonal action), $\mathcal{C}/F_k$ is a finite complex equipped with two tracks representing the splittings associated to the cyclic trees T_1 and T_2. Further, the projection maps $T_1 \leftarrow T_1 \times T_2 \to T_2$ descend to maps $T_1/F_k \leftarrow \mathcal{C}/F_k \to T_2/F_k$. The tracks in $\mathcal{C}/F_k$ are the preimages of the midpoints of the edges T_1/F_k and T_2/F_k.

Now to get a "subsurface" for a free factor $X \subset F_k$, we restrict the actions on T_1 and T_2 to the subgroup X and consider the core $\mathcal{C}^X \subset T_1^X \times T_2^X$. The natural inclusions $T_1^X \to T_1$ and $T_2^X \to T_2$ induce an inclusion $\mathcal{C}^X \to \mathcal{C}$ and a "subsurface inclusion" map $\mathcal{C}^X/X \to \mathcal{C}/X \to \mathcal{C}/F_k$. The key point is that $\mathcal{C}^X/X$ is a finite complex representing X. The picture one should keep in mind is the inclusion of the core of the cover of a subsurface into the cover associated to the subsurface, as well as its image in the surface under the covering map. See Figure 1.

Therefore, by analogy we should define the intersection number between a cyclic tree T_1 and a free factor X as the number of simply connected tracks associated to T_1 in $\mathcal{C}^X/X$. The map $\mathcal{C}^X/X \to T_1^X/X$ sends the simply connected tracks associated to T_1 to edges of T_1^X/X that have trivial edge stabilizer. Thus we define:

Definition 2.2 (Free volume)**.** Suppose X is a finitely generated free group that acts on a simplicial tree T such that the stabilizer of an edge is either trivial or cyclic. The *free volume* $\mathrm{vol}_T(X)$ of X with respect to T is the number of edges in the graph of groups decomposition T^X/X with trivial stabilizer.

A form of this definition appears in [16] in a more general setting. Notice that for a cyclic subgroup $X = \langle g \rangle$ we have $\mathrm{vol}_T(X) = \ell_T(g) = \#$ edges of T^X/X, as we desired for our notion of intersection. If T is equipped with a metric preserved by the action of X, the free volume $\mathrm{vol}_T(X)$ is the sum of lengths of the edges of T^X/X with trivial edge stabilizer. Clearly free volume only depends on the conjugacy class of the subgroup.

Lemma 2.3. *Let T be a cyclic tree for F_k and X a subgroup of F_k. If X is malnormal, then there is at most one edge of T^X/X that has a nontrivial stabilizer. Hence, if X is finitely generated, then:*

$$0 \leq \#\textit{edges of } T^X/X - \mathrm{vol}_T(X) \leq 1.$$

Proof. Suppose that T is the cyclic tree dual to the splitting of F_k over the cyclic subgroup $\langle c \rangle$. For any subgroup $X \subseteq F_k$, edges of T^X/X correspond to double cosets $Xg\langle c \rangle$ and the corresponding subgroup is represented by the X–conjugacy class of $X \cap g\langle c \rangle g^{-1}$ [34]. Suppose both $X \cap g\langle c \rangle g^{-1}$ and $X \cap h\langle c \rangle h^{-1}$ are nontrivial for some $g, h \in \langle c \rangle$. Thus for some $n \neq 0$, we have $gc^n g^{-1}, hc^n h^{-1} \in X$. Therefore, $gc^n g^{-1} \in X \cap gh^{-1}X(gh^{-1})^{-1}$. As X is malnormal, this implies $gh^{-1} \in X$ and therefore the double cosets $Xg\langle c \rangle$ and $Xh\langle c \rangle$ are the same. Hence, there is at most one edge of T^X/X with a nontrivial stabilizer. □

2.4. **Filling cyclic trees.** Recall that two simple closed curves α and β on a surface are said to *fill* when the sum of their intersection numbers with any arbitrary simple closed curve is positive. This naturally leads one to consider the following definition.

Definition 2.4 (Filling)**.** We say that two cyclic trees T_1 and T_2 for F_k *fill* if

$$\mathrm{vol}_{T_1}(X) + \mathrm{vol}_{T_2}(X) > 0 \qquad (\mathbf{F1})$$

for every proper free factor or cyclic subgroup $X \subset F_k$.

We thank Michael Handel for pointing out the follow lemma. The lemma says that the above condition (**F1**) is the minimal hypothesis one could impose on cyclic trees in Theorem 5.3.

Lemma 2.5. *Suppose T_1, T_2 are cyclic trees for F_k and δ_1, δ_2 are the respective Dehn twist (outer) automorphisms. Then T_1 and T_2 satisfy* (**F1**) *if and only if no conjugacy class of a proper free factor or cyclic subgroup of F_k is invariant under both δ_1 and δ_2.*

Proof. Suppose X is a cyclic subgroup or proper free factor of F_k such that $\mathrm{vol}_{T_i}(X) = 0$ for $i = 1$ or 2. Thus either X is contained in a vertex stabilizer of T_i, or else T_i^X/X is a single edge with a nontrivial stabilizer represented by an edge stabilizer of T_i. In the former case, it is clear that δ_i fixes the conjugacy class of X; in the latter case, conjugating δ_i by an appropriate inner automorphism results in an automorphism whose action on X agrees with the action of the Dehn twist automorphism of T^X. Thus the conjugacy class of X is invariant under δ_i. Hence if $\mathrm{vol}_{T_1}(X) + \mathrm{vol}_{T_2}(X) = 0$, then the conjugacy class of X is invariant under both δ_1 and δ_2.

For the converse, it follows from Theorem 4.6 that if X is a cyclic subgroup or proper free factor such that $\mathrm{vol}_{T_i}(X) \neq 0$ for $i = 1$ or 2, then the conjugacy class of X is not fixed by δ_i. Thus if $\mathrm{vol}_{T_1}(X) + \mathrm{vol}_{T_2}(X) > 0$, then the conjugacy class of X is not invariant under both δ_1 and δ_2. □

Now recall that for surfaces we have the following equivalent definitions of filling curves: (1) two curves fill if the complement of their union is a union of topological disks, and; (2) two curves fill if no proper subsurface contains the union of the curves. Each of these characterizations leads to an alternative notion for two cyclic trees T_1 and T_2 to fill:

(**F2**) F_k acts freely on the product $T_1 \times T_2$, i.e., no element of F_k fixes a point in each tree.
(**F3**) If c_1 fixes an edge of T_1 and c_2 fixes an edge of T_2, then the subgroup $\langle c_1, c_2 \rangle$ is not contained in a proper free factor of F_k.

The advantage of these alternate conditions is that (**F2**) can be checked using Stallings' graph pull-backs [37], and (**F3**) can be checked using a version of Whitehead's algorithm (see for instance [1] or [30]). Obviously (**F1**) implies (**F2**), and while some of the other relations are not clear, we will show that (**F2**) + (**F3**) implies (**F1**). In a later example we will see that (**F3**) is not implied by (**F1**) + (**F2**).

Proposition 2.6. *Suppose T_1 and T_2 are cyclic trees satisfying* (**F2**) *and* (**F3**)*. Then the trees T_1 and T_2 fill, i.e., T_1 and T_2 satisfy* (**F1**)*.*

Proof. As (**F2**) implies that no $g \in F_k$ fixes a point in both T_1 and T_2, clearly $\mathrm{vol}_{T_1}(\langle g \rangle) + \mathrm{vol}_{T_2}(\langle g \rangle) > 0$ for any $g \in F_k$.

Now suppose that X is a proper free factor such that $\mathrm{vol}_{T_1}(X) + \mathrm{vol}_{T_2}(X) = 0$. If X fixes a vertex in T_1 then X must act freely on T_2 by (**F2**) and hence $\mathrm{vol}_{T_2}(X) > 0$. Similarly $\mathrm{vol}_{T_1}(X) > 0$ if X fixes a vertex in T_2. Therefore we can assume that X does not fix a vertex in both T_1 and T_2. As X is a free factor and hence malnormal, by Lemma 2.3, the only way $\mathrm{vol}_{T_1}(X) + \mathrm{vol}_{T_2}(X) = 0$ is if both quotient graphs of groups T_1^X/X and T_2^X/X consist of a single edge with a nontrivial stabilizer. This contradicts (**F3**). Therefore $\mathrm{vol}_{T_1}(X) + \mathrm{vol}_{T_2}(X) > 0$ and hence T_1 and T_2 fill. □

We can use this Proposition to produce filling cyclic trees.

Example 2.7. Let T be the cyclic tree for F_3 dual to the splitting $F_3 = \langle a, c\rangle *_{\langle c\rangle} \langle b, c\rangle$ and let $\phi \in \mathrm{Out}\, F_3$ be the element represented by $a \mapsto b \mapsto c \mapsto ab$. We claim that the cyclic trees T and $T\phi^{-6}$ fill. For reference we make note of ϕ^6:

$$\phi^6: \quad \begin{aligned} a &\mapsto abbc \\ b &\mapsto bccab \\ c &\mapsto cababbc \end{aligned}$$

Vertex stabilizers of $T\phi^{-6}$ are conjugates of $\langle abbc, cababbc\rangle$ and $\langle bccab, cababbc\rangle$. Using pull-back diagrams it is easy to see that the intersections of the vertex stabilizers are empty. Hence the trees T and $T\phi^{-6}$ satisfy (**F2**) and therefore $\mathrm{vol}_T(\langle g\rangle) + \mathrm{vol}_{T\phi^{-6}}(\langle g\rangle) > 0$ for any $g \in F_k$.

Unfortunately, the trees T and $T\phi^{-6}$ do not satisfy (**F3**) as $\langle c, cababbc\rangle$ is a proper free factor of F_3 ($F_3 = \langle c, cababbc\rangle * \langle ab\rangle$). We can show that essentially this is the only such proper free factor, and that it satisfies (**F1**).

Suppose that X is a proper free factor that contains $\langle c_1, c_2\rangle$ where c_1 fixes an edge of T and c_2 fixes an edge of $T\phi^{-6}$. Then by replacing X by a conjugate, we can assume that $X = \langle c, g\phi^6(c)g^{-1}\rangle$ for some $g \in F_k$. However, it is easy to see that $\mathrm{vol}_T(X) \geq 3$ for this subgroup as the translation length of $\phi^6(c)$ in T is 4. Other proper free factors satisfy (**F1**) by the argument in Proposition 2.6. Hence $\mathrm{vol}_T(X) + \mathrm{vol}_{T\phi^{-6}}(X) > 0$ for any proper free factor and therefore T and $T\phi^{-6}$ fill.

To build filling cyclic trees in arbitrarily high rank we introduce two simplicial complexes naturally associated to F_k; these complexes appear in [24]. They are analogous to the curve complex for the mapping class group, i.e., the simplicial complex whose vertices are isotopy classes of simple closed curves and simplicies correspond to disjoint representatives.

The *dominance graph* $\mathcal{D}$ is the graph whose vertices correspond to conjugacy classes of proper free factors of F_k, where two such $[A]$ and $[B]$ are connected by an edge if there are representatives, $A' \in [A]$, $B' \in [B]$, with $A' \subset B'$ or $B' \subset A'$. This is the 1–skeleton of the free factor complex considered by Hatcher and Vogtmann [20].

We also consider the *cyclic splitting graph* $\mathcal{Z}'$, although what we actually require is the following variant of the like-named complex appearing in [24]: Vertices correspond to very small simplicial trees for F_k, i.e., simplicial trees T such that edge stabilizers are either trivial or maximal cyclic in adjacent vertex stabilizer, and the stabilizer of any tripod is trivial. Notice that cyclic trees where the edge stabilizers are generated by primitive elements (i.e., can be extended to a basis) are vertices in this graph. Two very small simplicial trees T_1 and T_2 are adjoined by an edge in $\mathcal{Z}'$ if there is a $g \in F_k$ such that $\ell_{T_1}(g) = \ell_{T_2}(g) = 0$, i.e., g fixes a point in both T_1 and T_2.

The following proposition should now be compared to the fact that two curves fill if and only if their distance in the curve complex is at least 3.

Proposition 2.8. *Suppose that T_1 and T_2 are cyclic trees with primitive cyclic edge generators c_1 and c_2, respectively, such that $d_{\mathcal{Z}'}(T_1, T_2) \geq 2$ and $d_{\mathcal{D}}([c_1], [c_2]) \geq 3$. Then the cyclic trees T_1 and T_2 fill.*

Proof. Since $d_{\mathcal{Z}'}(T_1, T_2) \geq 2$ there is no element $g \in F_k$ such that $\ell_{T_1}(g) = \ell_{T_2}(g) = 0$, hence the trees T_1 and T_2 satisfy (**F2**). Further since $d_{\mathcal{D}}([c_1], [c_2]) \geq 3$ there is no proper free factor $X \subset F_k$ or conjugates $c_1' \in [c_1]$ and $c_2' \in [c_2]$ such that $\langle c_1', c_2' \rangle \subseteq X$, hence the trees T_1 and T_2 satisfy (**F3**). Therefore by Proposition 2.6 the cyclic trees T_1 and T_2 fill. □

Remark 2.9. For $k \geq 3$, Kapovich and Lustig have shown that for a hyperbolic fully irreducible element $\phi \in \operatorname{Out} F_k$ and any two vertices $[A], [B] \in \mathcal{D}$ that $d_{\mathcal{D}}([A], \phi^n([B]))$ goes to infinity as $n \to \pm\infty$ [24]. Similarly for two vertices $T_1, T_2 \in \mathcal{Z}'$. Hence Proposition 2.8 shows for any cyclic tree T whose edge stabilizers are generated by conjugates of a primitive element and any hyperbolic fully irreducible element $\phi \in \operatorname{Out} F_k$, that given sufficiently large n, the pair T and $T\phi^n$ fill.

3. Computing free volume

In this section, we will explain how we use Stallings' folding to find the free volume of finitely generated subgroups of F_k relative to cyclic trees. This will be central to our proof of Theorem 4.6.

3.1. Cyclic splittings of F_k.

We begin by recalling two theorems which describe how any cyclic splitting of F_k must arise. For the case of amalgamations, we have the following theorem of Shenitzer:

Theorem 3.1 (Shenitzer [36]). *Suppose that F_k is expressed as an amalgamated free product $F_k = A *_{\langle c \rangle} B$. Then one of the following symmetric alternatives holds:*

(1) $A *_{\langle c \rangle} B = A *_{\langle c \rangle} \langle c, B_0 \rangle$ *with* $F_k = A * B_0$*; or*
(2) $A *_{\langle c \rangle} B = \langle A_0, c \rangle *_{\langle c \rangle} B$ *with* $F_k = A_0 * B$. □

Interchanging $A \leftrightarrow B$ we will always assume the first alternative holds. Consequently, a Dehn twist automorphism δ resulting from a splitting of F_k as an amalgamation over $\mathbb{Z}$ as above always arises as follows: There is a free splitting $F_k = A * B_0$ and an element $c \in A$ such that:

$$\forall a \in A \qquad \delta(a) = a$$
$$\forall b \in B_0 \qquad \delta(b) = cbc^{-1}.$$

A basis for F_k *relative to the cyclic tree* dual to $A *_{\langle c \rangle} B$ consists of the union of a basis for A and a basis for B_0.

There is an analogous theorem for HNN-extensions due to Swarup [39].

Theorem 3.2 (Swarup [39]). *Suppose that F_k is expressed as an HNN-extension $F_k = A*_{\mathbb{Z}}$. Express F in terms of A and an extra generator t, such that the edge group $\langle c \rangle = A \cap tAt^{-1}$. Then A has a free product* *structure $A = A_1 * A_2$, in such a way that one of the following symmetric alternatives holds:*

(1) $\langle c\rangle \subset A_1$, *and there exists* $a \in A$ *such that* $t^{-1}\langle c\rangle t = a^{-1}A_2 a$; *or*
(2) $t^{-1}\langle c\rangle t \subset A_1$, *and there exists* $a \in A$ *such that* $\langle c\rangle = a^{-1}A_2 a$. □

For alternative viewpoints and proofs see [4, 27, 38]. For our purposes we record the following restatement of Theorem 3.2.

Corollary 3.3. *Suppose that F_k is expressed as an HNN-extension $F_k = A*_{\mathbb{Z}}$. Then F_k has a free product decomposition $F_k = A_0 * \langle t_0\rangle$ and A has a free product decomposition $A = A_0 * \langle t_0^{-1}ct_0\rangle$ for some $c \in A_0$. Either $t = t_0 a$ (case (1) in Theorem 3.2), or $t = a^{-1}t_0^{-1}$ (case (2)).* □

Again, by interchanging $A \leftrightarrow tAt^{-1}$ we will always assume that first alternative holds. Thus any Dehn twist automorphism δ resulting from an HNN-extension over $\mathbb{Z}$ as above always arises as follows: There is a free splitting $F_k = A_0 * \langle t_0\rangle$ and an element $c \in A_0$ such that:

$$\forall a \in A_0 \qquad \delta(a) = a$$
$$\delta(t_0) = ct_0.$$

A basis for F_k *relative to the cyclic tree* dual to $A*_{\mathbb{Z}}$ consists of the union of a basis for A_0 and t_0.

Remark 3.4. We will usually restrict our attention to very small cyclic trees. This does not result in any loss of generality as any Dehn twist automorphism is a power of Dehn twist automorphism associated to a very small cyclic tree. Indeed, suppose T is a cyclic tree dual to an amalgamated free product that is not very small. By the above, this tree is dual to a splitting $A*_{\langle c^n\rangle}\langle c^n, B_0\rangle$ where $c \in A$ is an indivisible element. The associated Dehn twist is the n^{th} power of the Dehn twist associated to the very small cyclic tree dual to the splitting $A *_{\langle c\rangle} \langle c, B_0\rangle$.

3.2. **Free volume for an amalgamated free product.** Here we explain how to compute free volume for a finitely generated subgroup H with respect to a tree dual to an amalgamated product by associating a tree with a *free* F_k–action, using Shenitzer's Theorem. We consider a splitting of F_k as an amalgamated free product of the form:

$$F_k = A *_{\langle c\rangle} \langle c, B_0\rangle$$

with $F_k = A * B_0$ and $c \in A$ with c indivisible. Let $\mathcal{A} = \{a_1, \dots, a_j\}$ be a basis for A, and $\mathcal{B}_0 = \{b_{j+1}, \dots b_k\}$ a basis for B_0. We assume that c is cyclically reduced with respect to $\mathcal{A}$. Thus $\mathcal{A}\cup\mathcal{B}_0$ is a basis for F_k relative to T. Let $\Lambda = \Lambda_{\mathcal{A}\cup\mathcal{B}_0}$ be the k–rose labeled by the basis $\mathcal{A}\cup\mathcal{B}_0$. Then let $\Lambda_{\mathcal{A}}$ be the j–rose, labeled by the elements of $\mathcal{A}$, let $\Lambda_{\mathcal{B}_0}$ be the $(k-j)$–rose, labeled by the elements of $\mathcal{B}_0$, and let $\Lambda_{\mathcal{B}}$ be the $(k-j+1)$–rose resulting from wedging an additional circle corresponding to the element c to $\Lambda_{\mathcal{B}_0}$. There are natural inclusions $\iota_{\mathcal{A}}\colon \Lambda_{\mathcal{A}} \to \Lambda$ and $\iota_{\mathcal{B}_0}\colon \Lambda_{\mathcal{B}_0} \to \Lambda$ and an immersion

$\iota_{\mathcal{B}}\colon \Lambda_{\mathcal{B}} \to \Lambda$. We say that an edge of Λ corresponding to an element of $\mathcal{A}$ is an $\mathcal{A}$–edge and an edge of Λ corresponding to an element of $\mathcal{B}_0$ is a $\mathcal{B}_0$–edge.

Let $\tilde{\Lambda}_{\mathcal{A}}$ and $\tilde{\Lambda}_{\mathcal{B}}$ be the universal covers of $\Lambda_{\mathcal{A}}$ and $\Lambda_{\mathcal{B}}$ respectively. The covering maps naturally define immersions $\tilde{\iota}_{\mathcal{A}}\colon \tilde{\Lambda}_{\mathcal{A}} \to \Lambda$ and $\tilde{\iota}_{\mathcal{B}}\colon \tilde{\Lambda}_{\mathcal{B}} \to \Lambda$. Let $\mathcal{V}(\mathcal{A})$ denote the set of subtrees of $\tilde{\Lambda}$ which are lifts of $\tilde{\iota}_{\mathcal{A}}\colon \tilde{\Lambda}_{\mathcal{A}} \to \Lambda$ to $\tilde{\Lambda}$, and let $\mathcal{V}(\mathcal{B})$ denote the set of subtrees of $\tilde{\Lambda}$ which are lifts of $\tilde{\iota}_{\mathcal{B}}\colon \tilde{\Lambda}_{\mathcal{B}} \to \Lambda$ to $\tilde{\Lambda}$. In other words, $\mathcal{V}(\mathcal{A})$ is the set of minimal subtrees in $\tilde{\Lambda}$ for conjugates of A, and similarly for $\mathcal{V}(\mathcal{B})$.

Notice that the natural simplicial structure (i.e., vertices and edges) the trees in $\mathcal{V}(\mathcal{B})$ inherit from $\tilde{\Lambda}_{\mathcal{B}}$ is different from the induced simplicial structure on the trees when considered as subtrees of $\tilde{\Lambda}$, unless c is primitive and $c \in \mathcal{A}$. When we speak of "intersection" of these subtrees, we will usually refer to the natural inherited simplicial structure. To make this easier on the reader, we include the following definition.

Definition 3.5. Suppose X is a subtree of $\tilde{\Lambda}$ (considered with the induced simplicial structure) and L is a subtree in $\mathcal{V}(\mathcal{B})$ (considered with the inherited simplicial structure). We define the $\mathring{\cap}$–intersection of X and L, denoted $X \mathring{\cap} L$ by the following:

$$x \in X \mathring{\cap} L \Leftrightarrow \begin{array}{c} x \in e \subset X \cap L, \text{ where } e \text{ is a union of edges in } X \\ \text{and a union of edges of } L. \end{array}$$

There is an F_k–equivariant one-to-one correspondence between the set $\mathcal{V}(\mathcal{A}) \cup \mathcal{V}(\mathcal{B})$ and the set of vertices of T, defined by common stabilizer subgroups in F_k. Two vertices in T are adjacent if and only if the $\mathring{\cap}$–intersection of their corresponding subtrees in $\mathcal{V}(\mathcal{A})$ and $\mathcal{V}(\mathcal{B})$ is nonempty and hence an infinite line. Thus we have a description of T in terms of intersection of subtrees of $\tilde{\Lambda}$ associated to A and B.

Recall that H is a finitely generated subgroup of F_k, and that $\tilde{\Lambda}^H$ denotes the smallest H–invariant subtree of $\tilde{\Lambda}$. We seek to describe T^H/H (and hence compute $\mathrm{vol}_T(H)$) in terms of $\tilde{\Lambda}^H/H$ with additional data encoding the edge types. A subtree is *trivial* if it is a single vertex, otherwise it is *nontrivial*. We feature two sets of nontrivial subtrees of $\tilde{\Lambda}^H$:

(1) Nontrivial subtrees of the form $K^H = \tilde{\Lambda}^H \cap K$ for $K \in \mathcal{V}(\mathcal{A})$ which are not properly contained within a subtree $\tilde{\Lambda}^H \mathring{\cap} L$ for $L \in \mathcal{V}(\mathcal{B})$. We denote by $\mathcal{V}^H(\mathcal{A})$ the set of all such subtrees K^H.

(2) Nontrivial subtrees of the form $L^H = \tilde{\Lambda}^H \mathring{\cap} L$ for $L \in \mathcal{V}(\mathcal{B})$ which are not properly contained within a subtree $\tilde{\Lambda}^H \cap K$ for $K \in \mathcal{V}(\mathcal{A})$. We denote by $\mathcal{V}^H(\mathcal{B})$ the set of all such subtrees L^H.

Notice that $\mathcal{V}^H(\mathcal{A})$ is empty if and only if H is contained in a conjugate of B so that H fixes a vertex of T. Similarly, $\mathcal{V}^H(\mathcal{B})$ is empty if and only if H is contained a conjugate of A. Thus both $\mathcal{V}^H(\mathcal{A})$ and $\mathcal{V}^H(\mathcal{B})$ are empty if and only if H is contained in a conjugate of $\langle c\rangle$. In either of these cases the minimal tree T^H is a single point and $\mathrm{vol}_T(H) = 0$.

For each subtree $K^H \in \mathcal{V}^H(\mathcal{A})$ we have a corresponding vertex $v_K \in T$ (the vertex corresponding to $K \in \mathcal{V}(\mathcal{A})$, where $K \cap \tilde{\Lambda}^H = K^H$); denote the set of such vertices by $V^H(\mathcal{A})$. Likewise, for each component of $L^H \in \mathcal{V}^H(\mathcal{B})$ there is a corresponding vertex $v_L \in T$; denote the set of such vertices by $V^H(\mathcal{B})$. Note that this correspondence between components of $\mathcal{V}^H(\mathcal{A}) \cup \mathcal{V}^H(\mathcal{B})$ and vertices of T is H–equivariant as $\tilde{\Lambda}^H$ is H–equivariant.

Let $\mathcal{E}^H(\mathcal{A},\mathcal{B})$ denote the set of nonempty (but possibly trivial, i.e., a vertex) subtrees $K^H \cap L^H$ for $K^H \in \mathcal{V}^H(\mathcal{A})$ and $L^H \in \mathcal{V}^H(\mathcal{B})$. To each such subtree $K^H \cap L^H$ in $\mathcal{E}^H(\mathcal{A},\mathcal{B})$ is associated a (geometric) edge e_K^L in T, namely the edge with vertices v_K and v_L. Indeed as the corresponding subtrees $K \in \mathcal{V}(\mathcal{A})$ and $L \in \mathcal{V}(\mathcal{B})$ intersect (necessarily along an axis for some conjugate of c), the corresponding conjugates of A and B fix a common edge of T; this edge is e_K^L. We denote the set of such edges by $E^H(\mathcal{A},\mathcal{B})$. The correspondence between $\mathcal{E}^H(\mathcal{A},\mathcal{B})$ and $E^H(\mathcal{A},\mathcal{B})$ is of course H–equivariant.

Example 3.6. It is perhaps enlightening at this point to consider an example of the sets and subtrees described above. Let T be the cyclic tree dual to the splitting of $F_3 = \langle a_1, a_2, b\rangle$ as $\langle a_1, a_2\rangle *_{\langle a_1 a_2\rangle} \langle a_1 a_2, b\rangle$ and $H = \langle a_1 b\rangle$; the subtree $\tilde{\Lambda}^H$ is the axis of $a_1 b$. Then for $K \in \mathcal{V}(\mathcal{A})$, the subtrees $\tilde{\Lambda}^H \cap K$ are the edges of $\tilde{\Lambda}^H$ that are labeled by a_1. Similarly, the subtrees $\tilde{\Lambda}^H \cap L$ for $L \in \mathcal{V}(\mathcal{B})$ are the segments consisting of edges labeled by ba_1. With the inherited simplicial structure on a subtree L, only the edges on the axis of $a_1 b$ labeled by b are actually edges of L. Thus, for $L \in \mathcal{V}(\mathcal{B})$, the $\mathring{\cap}$–intersections, $\tilde{\Lambda}^H \mathring{\cap} L$ are the edges of $\tilde{\Lambda}^H$ labeled b. It is easy to see that $\mathcal{E}^H(\mathcal{A},\mathcal{B})$ is the set of vertices of $\tilde{\Lambda}^H$. See Figure 2 for corresponding edges $E^H(\mathcal{A},\mathcal{B})$. Notice that these edges form the axis of H in T, which is T^H.

Lemma 3.7. *Suppose H does not fix a point in T. Then the subcomplex in T consisting of vertices $V^H(\mathcal{A}) \cup V^H(\mathcal{B})$ and edges $E^H(\mathcal{A},\mathcal{B})$ is precisely the smallest H–invariant subtree T^H of T.*

Proof. Suppose that v_K and v_L are two vertices in $V^H(\mathcal{A}) \cup V^H(\mathcal{B})$. Then there exists an arc in $\tilde{\Lambda}^H$ which connects the component K to the component L. This arc passes through a sequence of subtrees $K = K_0, K_1, \ldots, K_n = L \in \mathcal{V}^H(\mathcal{A}) \cup \mathcal{V}^H(\mathcal{B})$. As the arc transitions from K_{i-1} to K_i, the intersections $K_{i-1} \cap K_i$ are non-empty and therefore correspond to edges $e_i = e_{K_{i-1}}^{K_i} \in E^H(\mathcal{A},\mathcal{B})$. By construction the edge path $e_1, \ldots, e_n$ connects v_K to v_L. Therefore the subcomplex consisting of vertices $V^H(\mathcal{A}) \cup V^H(\mathcal{B})$ and edges $E^H(\mathcal{A},\mathcal{B})$ is connected and hence an H–invariant subtree of T.

To prove minimality, note that every edge e in $E^H(\mathcal{A},\mathcal{B})$ lies on the axis of some element of H acting on T. Indeed, suppose e corresponds to $K \cap L \in \mathcal{E}^H(\mathcal{A},\mathcal{B})$ with $K \in \mathcal{V}^H(\mathcal{A})$ and $L \in \mathcal{V}^H(\mathcal{B})$. Since K is not contained in L there is an $x \in K - (K \cap L)$. Let $h \in H$ be such that the

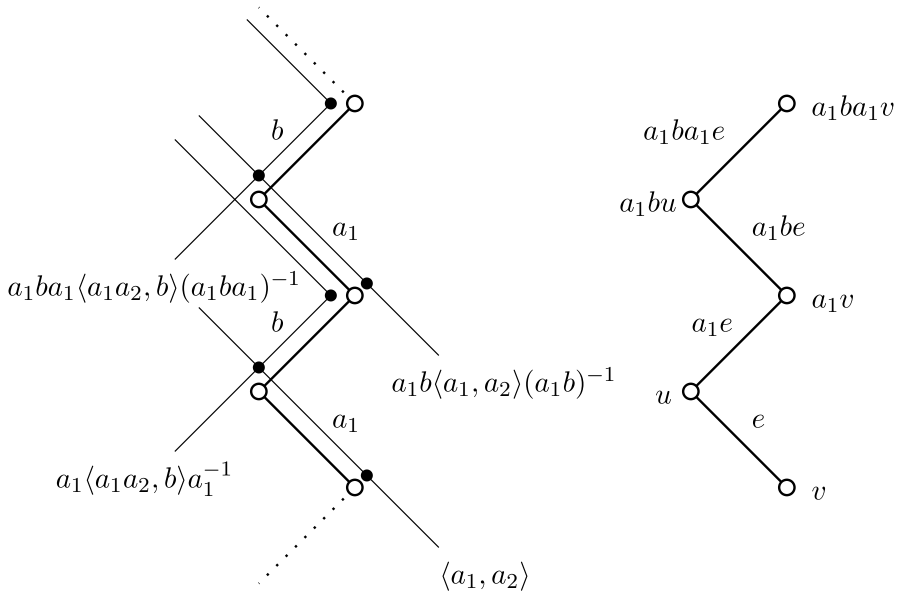

FIGURE 2. The edges $E^H(\mathcal{A}, \mathcal{B})$ for Example 3.6. On the left is the axis $\tilde{\Lambda}^H$ with the subtrees in $\mathcal{V}(\mathcal{A})$ and $\mathcal{V}(\mathcal{B})$ schematically draw in. The inherited simplicial structure on these subtrees is shown. On the right is the picture in T. The vertices u and v are stabilized by $\langle a_1, a_2\rangle$ and $\langle a_1a_2, b\rangle$, respectively, and the edge e is stabilized by $\langle a_1a_2\rangle$.

edge path from x to hx is contained in the axis of h and the edge path from x to hx contains $K \cap L$. Such an element exists since the action of H on $\tilde{\Lambda}^H$ is minimal. Moreover, by choosing the axis of h to be "complicated enough" with respect to $\mathcal{V}(\mathcal{A})$ and $\mathcal{V}(\mathcal{B})$, i.e., the axis intersects several of these subtrees, we can assume that h does not fix a point of T. Notice that the axis of h in T contains e. It is well-known that when a group acts on a tree without a global fixed point, the minimal tree is precisely the union of the axes of its elements [11]. □

We introduce some terminology which will be useful for classifying the subtrees in $\mathcal{V}^H(\mathcal{A})$, $\mathcal{V}^H(\mathcal{B})$, and $\mathcal{E}^H(\mathcal{A}, \mathcal{B})$. Fix an immersion $\gamma\colon [0,1] \to \Lambda$ that factors through $[0,1] \to S^1 \to \Lambda$, where the first map identifies 0 and 1, and the second map represents the conjugacy class of $c \in F_k \cong \pi_1(\Lambda)$. We let Λ^H be the graph $\tilde{\Lambda}^H/H$. A *chain* is an ordered set $\alpha = (\gamma_1, \ldots, \gamma_\ell)$, where γ_i is a lift of γ to Λ^H, with $\gamma_i(1) = \gamma_{i+1}(0)$ for $i = 1, \ldots, \ell - 1$. The *vertices* of a chain are $\mathcal{V}(\alpha) = \gamma_1(0) \cup \bigcup_{i=1}^{\ell} \gamma_i(1)$. Notice that vertices of a chain are vertices of Λ^H, but vertices contained in the image of α are not necessarily vertices of the chain. We often identify a chain with its image in Λ^H.

We refer to an edge in Λ^H as an $\mathcal{A}$–edge or $\mathcal{B}_0$–edge according to its image in Λ. A chain α is *nonessential* if

(1) any edge adjacent to α is a $\mathcal{B}_0$–edge which is adjacent to α at a vertex in $\mathcal{V}(\alpha)$; or
(2) the only edges adjacent to α are $\mathcal{A}$–edges.

Otherwise we say α is *essential*. In other words, α is essential only if it is adjacent to a $\mathcal{B}_0$–edge at a vertex of Λ^H that is not a vertex of α, or it is adjacent to both an $\mathcal{A}$–edge and a $\mathcal{B}_0$–edge. The edges in a nonessential chain adjacent only to $\mathcal{B}_0$–edges are considered $\mathcal{B}_0$–edges. The set of all maximal essential chains in Λ^H is denoted by $\alpha(\Lambda^H)$.

We say a vertex is *essential* if it is not a chain vertex of any essential chain and it is adjacent to both an $\mathcal{A}$–edge and a $\mathcal{B}_0$–edge. The set of all essential vertices we denote by $\mathcal{V}_{ess}(\Lambda^H)$.

Lemma 3.8. *With the notation above, the image of a subtree in $\mathcal{E}^H(\mathcal{A}, \mathcal{B})$ in Λ^H is either a maximal essential chain or an essential vertex. Conversely, every maximal essential chain or essential vertex is the image of some subtree in $\mathcal{E}^H(\mathcal{A}, \mathcal{B})$.*

Proof. Let $K \in \mathcal{V}^H(\mathcal{A})$ and $L \in \mathcal{V}^H(\mathcal{B})$ and suppose $K \cap L$ is nonempty. First suppose $K \cap L$ is a vertex. Hence its image in Λ^H is adjacent to both an $\mathcal{A}$–edge and a $\mathcal{B}_0$–edge. Furthermore it is not the vertex of a chain as such a chain would lift to a segment in $\tilde{\Lambda}^H$ adjacent to this vertex and contained in both K and L, contradicting the fact that their intersection is a single vertex. Hence the image of $K \cap L$ is an essential vertex.

Now suppose $K \cap L$ is a nondegenerate segment. Its image in Λ^H is clearly a maximal chain. Furthermore, as K is not contained in L, and L is not contained in K, the chain is essential.

For the converse, we show how to find the subtrees K and L. Let Λ_A be the complement in Λ^H of the union of the interiors of the $\mathcal{B}_0$–edges. There is exactly one component of Λ_A that contains the given maximal essential chain or essential vertex. Let K be a lift of this component to $\tilde{\Lambda}^H$, and notice that $K \in \mathcal{V}^H(\mathcal{A})$. Similarly, let Λ_{B_0} be the complement in Λ^H of the union of the interior of the $\mathcal{A}$–edges. Attach each chain in $\alpha(\Lambda^H)$ to Λ_{B_0} along its vertices to the appropriate component and call the resulting set of components Λ_B. Again, there is exactly one component of Λ_B that contains the given maximal essential chain or vertex. Let L be a lift of this component to $\tilde{\Lambda}^H$ that intersects K, and notice that $L \in \mathcal{V}^H(\mathcal{B})$. The given maximal essential chain or essential vertex is the image of $K \cap L$. □

By construction, two edges $e_{K_1}^{L_1}$ and $e_{K_2}^{L_2}$ in T^H are identified by $h \in H$ if and only if $h^{\pm 1}(K_1 \cap L_1) = K_2 \cap L_2$. Hence edges of T^H/H correspond to maximal essential chains and essential vertices in Λ^H. Furthermore, as the action of $\tilde{\Lambda}^H$ is free, an edge e_K^L has a nontrivial edge stabilizer if and only if $K \cap L$ is an infinite line, in which case the corresponding essential chain in Λ^H has two vertices that are identified. We say that an essential chain α in Λ^H is *simply connected* if the elements of $\mathcal{V}(\alpha)$ are all distinct. As edge stabilizers are maximal cyclic subgroups, an edge in $E^H(\mathcal{A}, \mathcal{B})$ has nontrivial stabilizer if and only if it corresponds to a non-simply connected chain. The subset of simply connected maximal essential chains is denoted $\alpha_{sc}(\Lambda^H)$.

We have now proved:

Theorem 3.9. *Suppose that T is a very small cyclic tree dual to a splitting $F_k = A *_{\langle c \rangle} \langle c, B_0 \rangle$ and H is a finitely generated subgroup of F_k. Let $\mathcal{A} \cup \mathcal{B}_0$ be a basis relative to T, and define $\Lambda = \Lambda_{\mathcal{A} \cup \mathcal{B}_0}$ and $\Lambda^H = \tilde{\Lambda}^H / H$. Then:*

$$\text{vol}_T(H) = \#|\alpha_{sc}(\Lambda^H)| + \#|\mathcal{V}_{ess}(\Lambda^H)|.$$

□

Example 3.10. Let T be the cyclic tree dual to the splitting $F_3 = \langle a, b \rangle *_{[a,b]} \langle [a, b], c \rangle$. Then the basis $\{a, b\} \cup \{c\}$ is relative to this splitting. Let H be a subgroup in the conjugacy class represented by the graph in Figure 3. Chains are denoted by dotted lines, all of which are essential, and essential vertices are black. There are two simply connected chains and nine essential vertices; hence $\text{vol}_T(H) = 11$. In Figure 4 we demonstrate the vertex groups of the induced graph of groups decomposition T^H/H. The underlying graph of T^H/H has three vertices: v_1, v_2 and v_3. There are seven edges from v_1 to v_2 and five edges from v_2 to v_3, one of which has a nontrivial stabilizer.

We state one final definition which will be used in Section 4.

Definition 3.11. Let H and Λ^H be as in Theorem 3.9. A vertex of Λ^H is a *crossing vertex* if it is either essential, or if it is a vertex of an essential chain and is adjacent to a $\mathcal{B}_0$–edge.

3.3. **Free volume for an HNN-extension.** Now suppose that we have a cyclic HNN-extension

$$F_k = (A_0 * \langle t_0^{-1} c t_0 \rangle) *_{\langle c \rangle}$$

as in Corollary 3.3, with $c \in A_0$ indivisible and T a cyclic tree. Let $\mathcal{A}_0 = \{a_1, \dots, a_{k-1}\}$ be a basis for A_0. Then $\mathcal{A}_0 \cup \{t_0\}$ is a basis for F_k relative to T. Let $\Lambda_{\mathcal{A}_0}$ be the $(k-1)$–petaled rose labeled by the elements of $\mathcal{A}_0$, and let $\Lambda = \Lambda_{\mathcal{A}_0 \cup \{t_0\}}$ be the k–petaled rose labeled by the basis $\mathcal{A}_0 \cup \{t_0\}$. There is a natural inclusion $\iota_{\mathcal{A}_0} : \Lambda_{\mathcal{A}_0} \to \Lambda$ which lifts to an immersion $\tilde{\iota}_{\mathcal{A}_0} : \tilde{\Lambda}_{\mathcal{A}_0} \to \Lambda$. Now let $\Lambda_{\mathcal{A}}$ be the k–rose, labeled by the elements of $\mathcal{A}_0 \cup \{t_0 c t_0^{-1}\}$. There is a natural map $\iota_{\mathcal{A}} \colon \Lambda_{\mathcal{A}} \to \Lambda$ which lifts to a map $\tilde{\iota} : \tilde{\Lambda}_{\mathcal{A}} \to \Lambda$ from the universal cover of $\Lambda_{\mathcal{A}}$. As before, we say that an edge of Λ corresponding to an element of $\mathcal{A}_0$ is an $\mathcal{A}_0$–edge, and that an edge of Λ corresponding to t_0 is a $\{t_0\}$–edge. A $\{t_0\}$–edge is positively oriented if it corresponds to t_0 and negatively oriented if it corresponds to t_0^{-1}.

Let $\mathcal{V}(\mathcal{A})$ be the set of lifts of $\tilde{\iota} : \tilde{\Lambda}_{\mathcal{A}} \to \Lambda$ to $\tilde{\Lambda}$. Each lift corresponds uniquely to a vertex of T, and two vertices are adjacent if their

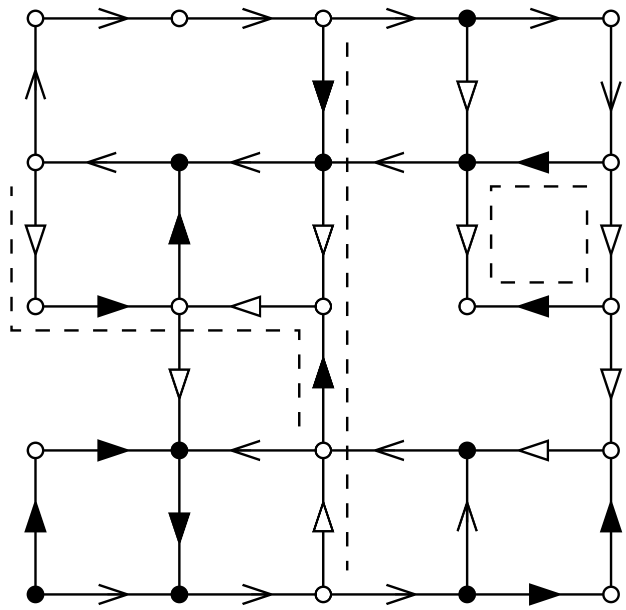

FIGURE 3. The graph Λ^H in Example 3.10. The arrows describe the immersion $\Lambda^H \to \Lambda$. The black arrows are sent to the petal corresponding to "a", the white arrows to "b" and the open arrows to "c".

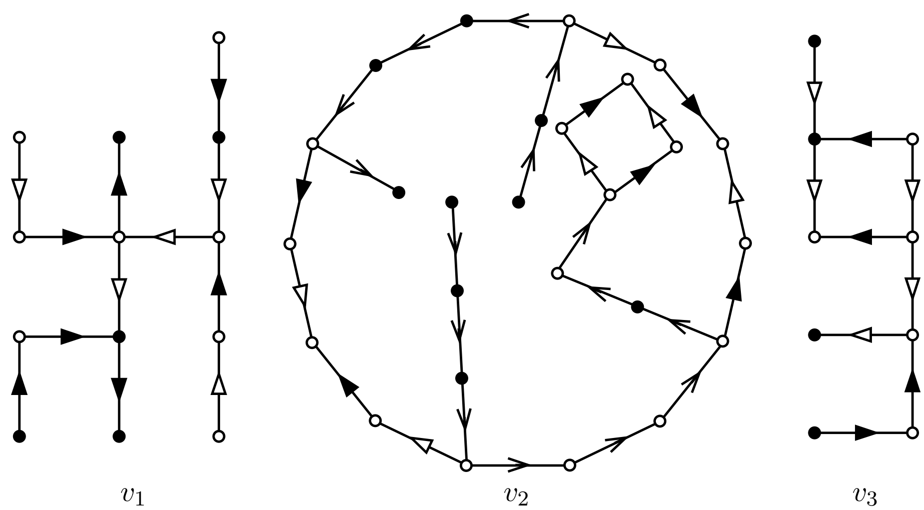

FIGURE 4. Graphs representing the conjugacy class of the vertex groups of the graph of groups decomposition T^H/H in Example 3.10.

$\mathring{\cap}$–intersection of the two corresponding subtrees of $\tilde{\Lambda}$ is nonempty and hence an infinite line. Let $\mathcal{E}(\mathcal{A})$ denote the set of all such pairwise $\mathring{\cap}$–intersections between elements of $\mathcal{V}(\mathcal{A})$. Let H be a finitely generated subgroup of F_k, and let $\tilde{\Lambda}^H$ be its minimal subtree in $\tilde{\Lambda}$. We denote by $\mathcal{V}^H(\mathcal{A})$ the set consisting of nontrivial subtrees of the form $K^H = \tilde{\Lambda}^H \mathring{\cap} K$ for $K \in \mathcal{V}(\mathcal{A})$ which are not properly contained in a subtree $\tilde{\Lambda}^H \mathring{\cap}$ for any other $L \in \mathcal{V}(\mathcal{A})$. We then let $\mathcal{E}^H(\mathcal{A})$ denote the set of (possibly trivial) subtrees $K^H \cap L^H$ of trees K^H and L^H in $\mathcal{V}^H(\mathcal{A})$. Lemma 3.7 transfers readily to the HNN-case, and so we have a hold on the minimal subtree T^H.

A chain in Λ^H is defined as in the amalgamated setting for the conjugacy class of $c \in F_k$. We define vertices of a chain and simple connectivity of chain as before.

We refer to an edge in Λ^H as an $\mathcal{A}_0$–edge or $\{t_0\}$–edge according to its image in Λ. A chain α is *nonessential* if:

(1) any edge adjacent to α is a positively oriented $\{t_0\}$–edge which is adjacent to α at a vertex in $\mathcal{V}(\alpha)$; or
(2) α is only adjacent to $\mathcal{A}_0$–edges and negatively oriented $\{t_0\}$–edges.

Otherwise we say that α is *essential*. As in the case of amalgamated free products, the positively oriented $\{t_0\}$–edges adjacent to a nonessential chain are considered $\mathcal{A}_0$–edges. The set of all maximal essential chains on Λ^H is denoted by $\alpha(\Lambda_H)$. The subset of simply connected essential chains is denoted $\alpha_{sc}(\Lambda_H)$.

We say that a vertex is *essential* if it is the initial vertex of a positively oriented $\{t_0\}$–edge, but is not a chain vertex of any chain. The set of all essential vertices we denote by $\mathcal{V}_{ess}(\Lambda_H)$.

With these definitions in place, we give an analogue of Lemma 3.8 whose proof is similar.

Lemma 3.12. *With the notation above, the image of a subtree in $\mathcal{E}^H(\mathcal{A})$ in Λ^H is either a maximal essential chain or an essential vertex. Conversely, every maximal essential chain or vertex is the image of some subtree in $\mathcal{E}^H(\mathcal{A})$.*

We can now state how to count free volume for a finitely generated subgroup with respect to a cyclic tree dual to an HNN-extension, as the argument now proceeds as for the amalgamation case.

Theorem 3.13. *Suppose that T is a very small cyclic tree dual to a splitting $F_k = (A_0 * \langle t_0 c t_0^{-1} \rangle) *_{\langle c \rangle}$ and H is a finitely generated subgroup of F_k. Let $\mathcal{A}_0 \cup \{t_0\}$ be a basis relative to T, and define $\Lambda = \Lambda_{\mathcal{A}_0 \cup \{t_0\}}$ and $\Lambda^H = \tilde{\Lambda}^H / H$. Then:*

$$\mathrm{vol}_T(H) = \#|\alpha_{sc}(\Lambda^H)| + \#|\mathcal{V}_{ess}(\Lambda^H)|.$$

□

Example 3.14. Here we let T be the cyclic tree dual to the splitting $F_3 = \langle a, b, t_0^{-1}[a,b]t_0 \rangle *_{\langle [a,b] \rangle}$, with cyclic edge generator $c = [a,b]$. Let H be a subgroup in the conjugacy class represented by the graph in Figure 5. The eight chains are indicated by dotted lines; three of these are inessential, and

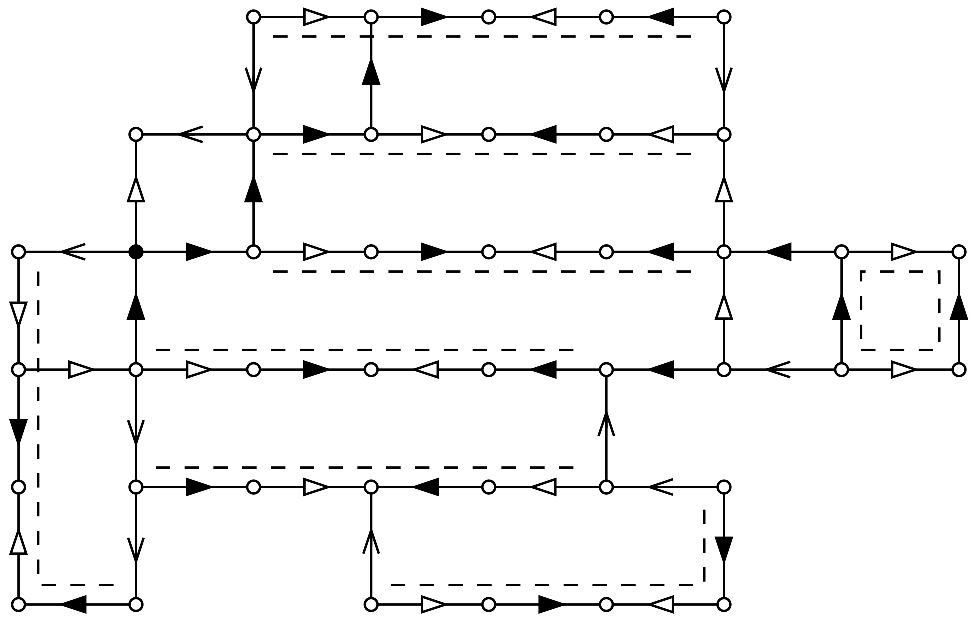

FIGURE 5. The graph Λ^H in Example 3.14. The arrows describe the immersion $\Lambda^H \to \Lambda$. The black arrows are sent to the petal corresponding to "a", white arrows to "b", and the open arrows to "t_0". Chains are indicated by dotted line segments.

one is not simply connected. There is a single essential vertex, indicated in black. The free volume is therefore $\mathrm{vol}_T(H) = 5$.

Again we have a notion of crossing vertex for an HNN-extension similar to Definition 3.11.

Definition 3.15. Let H and Λ^H be as in Theorem 3.13. A vertex of Λ^H is a *crossing vertex* if it is an essential vertex, or if it is a vertex of an essential chain and is adjacent to a positively oriented $\{t_0\}$–edge.

4. Twisted volume growth

Let T_1 and T_2 be two very small cyclic trees for F_k with edge stabilizers respectively generated by conjugates of the elements c_1 and c_2 and with associated Dehn twist elements δ_1 and δ_2. Fix bases $\mathcal{T}_1 = \mathcal{A}_1 \cup \mathcal{B}_1$ and $\mathcal{T}_2 = \mathcal{A}_2 \cup \mathcal{B}_2$ for F_k relative to these trees. Let $\Lambda_1 = \Lambda_{\mathcal{T}_1}$ and $\Lambda_2 = \Lambda_{\mathcal{T}_2}$ be the k–petaled roses for these bases, as constructed in Section 3.

The goal of this section is to prove Theorem 4.6 of the introduction; that is, we want to find bounds for $\mathrm{vol}_{T_2}(\delta_1^{\pm n}(H))$ when H is a finitely generated malnormal or cyclic subgroup. To begin, we discuss how the graph of groups decomposition described in Section 3 of a finitely generated subgroup H and the according free volume of H changes upon twisting.

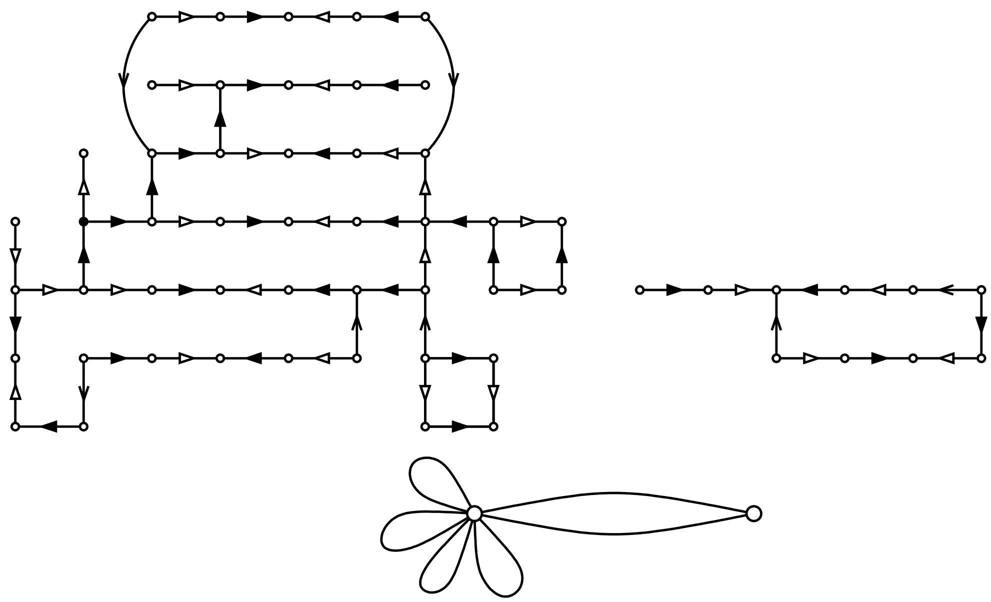

Figure 6. The top two graphs represent the conjugacy class of the vertex groups of the graph of groups decomposition T^H/H in Example 3.14. The graph below represents the graph of groups T^H/H.

4.1. **Graph composition.** Let $\nu\colon \Lambda_1 \to \Lambda_2$ be a (linear) homotopy equivalence representing the change in marking. Suppose $\rho\colon \mathcal{H} \to \Lambda_1$ is a map (not necessarily an immersion) such that the image of $\pi_1(\mathcal{H})$ in $\pi_1(\Lambda_1)$ is a conjugate of H. Then we can form the composition $\nu \circ \rho\colon \mathcal{H} \to \Lambda_2$. We define $\mathcal{H}_{\Lambda_2}$ as the graph (equipped with the map $\rho_{\Lambda_2}\colon \mathcal{H}_{\Lambda_2} \to \Lambda_2$) obtained from $\mathcal{H}$ by subdividing each edge $e \subset \mathcal{H}$ so that every component of the pre-image of the vertex in Λ_2 is a vertex. We say that $\mathcal{H}_{\Lambda_2}$ is obtained from $\mathcal{H}$ by *graph composition using* ν.

The following lemma is clear from the definitions.

Lemma 4.1. *After folding and pruning the map* $\rho_{\Lambda_2}\colon \mathcal{H}_{\Lambda_2} \to \Lambda_2$ *we obtain an immersion* $\rho_2^H\colon \mathcal{G}_2^H \to \Lambda_2$ *of a core graph* $\mathcal{G}_2^H$ *for the subgroup* H. $\square$

4.2. **Graph surgery.** Fix an immersion of a core graph $\rho_1^H\colon \mathcal{G}_1^H \to \Lambda_1$. We label edges as $\mathcal{A}_1$–edges or $\mathcal{B}_1$–edges according to their image in Λ_1, as in Section 3. We then locate the simply connected and non-simply connected chains, and the essential and nonessential vertices and chains. Recall that a *crossing vertex* is described in Definitions 3.11 and 3.15

For $n \geq 0$, let $a_n = [0,1]$ be an interval subdivided into $|c_1^n|_{\mathcal{T}_1}$ edges and let $\bar{a}_n$ denote a_n with opposite orientation. Let $v \in \mathcal{G}_1^H$ be a crossing vertex. Add a new vertex v' and insert a copy of the the interval a_n by attaching the vertex 0 to v and the vertex 1 to v'. Now perform one of the two following operations:

(1) If T_1 is dual to an amalgamated free product, then for each $\mathcal{B}_1$–edge e adjacent to v, redefine the initial vertex of e to be v'.
(2) If T_1 is dual to an HNN-extension (so that $\mathcal{B}_1$ is equal to the one-element set $\{t_0\}$ for some t_0), then redefine to be v' the initial vertex of the unique positively oriented $\{t_0\}$–edge adjacent to v.

Let Υ_n^H be the graph obtained by performing the above appropriate operation at each crossing vertex of $\mathcal{G}_1^H$. Define a map $\rho_1\colon \Upsilon_n^H \to \Lambda_1$ which is equal to ρ_1^H on edges of $\mathcal{G}_1^H$, and which maps each new arc a_n to the edge path for c_1^n in Λ_1. We say that Υ_n^H is obtained from $\mathcal{G}_1^H$ by *graph surgery along* T_1.

Lemma 4.2. *After folding and pruning the map $\rho_1\colon \Upsilon_n^H \to \Lambda_1$, we obtain the immersion of the core graph $\rho_1^{\delta_1^n(H)}\colon \mathcal{G}_1^{\delta_1^n(H)} \to \Lambda_1$ for the subgroup $\delta_1^n(H)$.*

Proof. Let ν and $\mathcal{H}$ be as in Section 4.1, where Λ_2 is the k–petaled rose corresponding to the image of the basis $\mathcal{A}_1 \cup \mathcal{B}_1$ under the Dehn twist δ_1. Recall that this means that the petals of Λ_2 correspond to elements of the basis $\mathcal{A}_1 \cup c_1\mathcal{B}_1 c_1^{-1}$ if T_1 is dual to an amalgamated free product, and to the basis $\mathcal{A}_1 \cup c_1\mathcal{B}_1$ if T_1 is dual to an HNN-extension. Also as in Section 4.1, let $\mathcal{H}_{\Lambda_2}$ be the graph obtained from $\mathcal{H}$ by graph composition using ν. If T_1 is dual to an HNN-extension, then $\mathcal{H}_{\Lambda_2}$ is equal to Υ_n^H. Otherwise the graph Υ_n^H is obtained from $\mathcal{H}_{\Lambda_2}$ by folding and pruning segments corresponding to $\bar{a}_n a_n$ between adjacent $\mathcal{B}_1$–edges. □

It is clear that by inserting $\bar{a}_n$ at each crossing vertex to obtain Υ_{-n}^H, we can fold and prune to obtain an immersion of a core graph $\mathcal{G}_1^{\delta_1^{-n}(H)}$ for the subgroup $\delta_1^{-n}(H)$.

Notice that if the crossing vertex v lies on a non-simply connected chain, then the entire newly added interval a_n can be folded onto this chain; it is for this reason that we record free volume instead of total volume. Combining Lemmas 4.1 and 4.2 we obtain the following corollary describing the change in the graph of groups decomposition for H upon twisting.

Corollary 4.3. *Suppose $\rho_1^H\colon \mathcal{G}_1^H \to \Lambda_1$ is an immersion of a core graph for H and let $\rho_1\colon \Upsilon_N^H \to \Lambda_1$ be the result obtained by graph surgery along T_1. Then after folding and pruning the composition $\nu \circ \rho_1\colon \Upsilon_n^H \to \Lambda_2$, we obtain an immersion $\rho_2^{\delta_1^n(H)}\colon \mathcal{G}_2^{\delta_1^n(H)} \to \Lambda_2$ of a core graph $\mathcal{G}_2^{\delta_1^n(H)}$ for the subgroup $\delta_1^n(H)$.* □

In the next section we show how to control the amount of folding and pruning that takes place on the newly added intervals a_n in the above corollary.

4.3. **Safe essential pieces.** Suppose that T_2 is a very small cyclic tree dual to an amalgamated free product. By conjugating the basis $\mathcal{T}_1$ (so that it remains a basis relative to T_1 and so that the associated Dehn twist automorphism defines the same outer automorphism class), we can assume that c_1 is cyclically reduced with respect to $\mathcal{T}_2$. Moreover, if c_1 does not fix a point in T_2, then by further conjugating we can assume that, as a reduced word in $\mathcal{T}_2$, the element c_1 has the form:

$$c_1 = x_1 c_2^{i_1} y_1 c_2^{j_1} \cdots x_m c_2^{i_m} y_m c_2^{j_m} \tag{4.1}$$

where for $r = 1, \ldots, m$, the word y_r is a nontrivial word in $\mathcal{B}_2$ and the word x_r is a nontrivial word in $\mathcal{A}_2$ such that zx_r and $x_r z$ are reduced for $z = c_2, c_2^{-1}$. (This last statement requires the adjective very small.) Thus $|c_1^n|_{\mathcal{T}_2} = n|c_1|_{\mathcal{T}_2}$ and $\ell_{T_2}(c_1^n) = 2mn$.

Now suppose that T_2 is a very small cyclic tree for an HNN-extension. Again by conjugating the basis $\mathcal{T}_1$, we can assume that c_1 is cyclically reduced with respect to $\mathcal{T}_2$. Moreover, if c_1 does not fix a point in T_2, then by further conjugating, we can assume that as a reduced word in $\mathcal{T}_2$, the element c_1 has the form:

$$c_1 = x_1(c_2^{i_1} t_0)^{\epsilon_1} x_2 (c_2^{i_2} t_0)^{\epsilon_2} \cdots x_m (c_2^{i_m} t_0)^{\epsilon_m}$$

where for $r = 1, \ldots, m$, the word x_r is a (possibly trivial) word in $\mathcal{A}_2 \cup \{t_0^{-1} c_2 t_0\}$, where $\epsilon_r \in \{\pm 1\}$; and if $\epsilon_r = 1$, then $x_r z$ is a reduced word for $z = c_2, c_2^{-1}$, and if $\epsilon_r = -1$, then zx_{r+1} is a reduced word for $z = c_2, c_2^{-1}$, where the subscript is considered modulo m. (Again, this last statement requires the adjective very small.) Thus $|c_1^n|_{\mathcal{T}_2} = n|c_1|_{\mathcal{T}_2}$ and $\ell_{T_2}(c_1^n) = mn$.

In either of two above cases, we say that c_1 is T_2*–reduced*. For the remainder of this section, we will always assume that c_1 is T_2–reduced.

Let $\alpha^n_{\Lambda_2} = [0,1]$ be the interval subdivided into $|c_1^n|_{\mathcal{T}_2}$ edges. There is a map $\alpha^n_{\Lambda_2} \to \mathcal{G}_2^{\langle c_1^n \rangle} \to \Lambda_2$, where the first map identifies the endpoints of $\alpha^n_{\Lambda_2}$, and the second map is the immersion of the core graph whose image represents the conjugacy class of c_1^n. As c_1 is cyclically reduced with respect to $\mathcal{T}_2$, no folding takes place after identifying the vertices of $\alpha^n_{\Lambda_2}$. Also, as c_1 is T_2–reduced, essential chains and essential vertices relative to the basis $\mathcal{T}_2$ can be considered as subsets of $\alpha^n_{\Lambda_2}$. These essential chains and essential vertices are referred to as *essential pieces* (relative to $\mathcal{T}_2$).

We say that an essential piece in $\alpha^n_{\Lambda_2}$ is *safe* if the vertex or chain does not intersect a vertex of one of the extremal $BCC(\mathcal{T}_1, \mathcal{T}_2)$ edges of $\alpha^n_{\Lambda_2}$. It is clear that at most $2BCC(\mathcal{T}_1, \mathcal{T}_2) + 2$ essential pieces in $\alpha^n_{\Lambda_2}$ are not safe.

Example 4.4. Let T_2 be the cyclic tree dual to the splitting $F_3 = \langle a, c\rangle *_{\langle c \rangle} \langle c, b\rangle$. Suppose T_1 is another cyclic tree such that $c_1 = ababac^3b$ (this is T_2–reduced) and $BCC(\mathcal{T}_1, \mathcal{T}_2) = 3$. The segment $\alpha^1_{\Lambda_2}$ is shown in Figure 7. The only safe essential piece is the fifth from the left essential vertex.

Consider an immersion of a core graph $\rho\colon \mathcal{G}_1^H \to \Lambda_1$. The image of a chain $\alpha = (\gamma_1, \ldots, \gamma_\ell) \in \alpha(\mathcal{G}_1^H)$ in $(\mathcal{G}_1^H)_{\Lambda_2}$, the graph composition of $\mathcal{G}_1^H$ using $\nu\colon \Lambda_1 \to \Lambda_2$, is naturally identified with a copy of the segment $\alpha^\ell_{\Lambda_2}$.

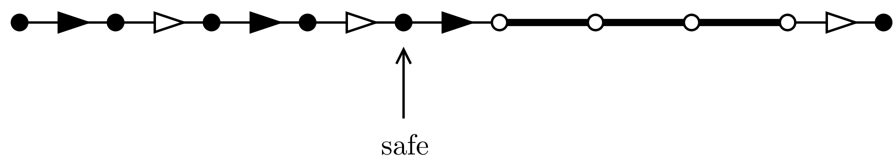

FIGURE 7. The segment $\alpha^1_{\Lambda_2}$ for T_2 in Example 4.4. The black arrows are sent to the petal corresponding to "a", white arrows to "b" and the thick line without arrows represents an essential chain. Essential vertices are black.

To obtain the inequality of Theorem 4.6, we determine the number safe pieces resulting from twisting which contribute to new volume. Upon twisting, safe essential pieces might get folded with surgered segments and then pruned. We account for these pruned safe pieces by showing that they must contribute to the original free volume of H with respect to T_2. This is the use of the following proposition.

Proposition 4.5. *Suppose that H is a finitely generated malnormal or cyclic subgroup of F_k where $\text{rank}(H) \leq R$. Then there is an $M = M(R)$, such that given $\rho\colon \mathcal{G}^H_1 \to \Lambda_1$, an immersion of the core graph $\mathcal{G}^H_1$, we have:*

$$\sum_{(\gamma_1,\ldots,\gamma_\ell)\in\alpha(\mathcal{G}^H_1)} \#\textit{safe essential pieces in } \alpha^\ell_{\Lambda_2} \leq M(\text{vol}_{T_2}(H)+1). \tag{4.2}$$

Proof. Recall that $\mathcal{G}^H_2$ is the core graph obtained from folding and pruning the graph $\rho_{\Lambda_2}\colon (\mathcal{G}^H_1)_{\Lambda_2} \to \Lambda_2$ obtained in Lemma 4.1. As the subgroup H is malnormal or cyclic, it follows from Lemma 2.3 that the number of edges in the graph T^H_2/H exceeds $\text{vol}_{T_2}(H)$ by at most one. Therefore from the discussions in Sections 3.2 and 3.3, it follows that there is at most one essential chain of $\mathcal{G}^H_2$ relative to $\mathcal{T}_2$ which is not simply connected and hence does not contribute to $\text{vol}_{T_2}(H)$. We will thus demonstrate (4.2) by showing that the sum on the left hand side of the inequality is less than M times the number of essential chains and vertices in $\mathcal{G}^H_2$.

Let $\alpha = (\gamma_1, \ldots, \gamma_\ell)$ be a chain in $\mathcal{G}^H_1$. As c_1 is T_2–reduced and by bounded cancellation, any safe essential piece in $\alpha^\ell_{\Lambda_2}$ survives as a subset after folding and pruning $(\mathcal{G}^H_1)_{\Lambda_2}$ to get $\mathcal{G}^H_2$. What needs to be shown is that any such safe essential piece of a chain in $\mathcal{G}^H_1$ is part of an essential piece of $\mathcal{G}^H_2$ and that over all chains in $\alpha(\mathcal{G}^H_1)$, only boundedly many safe pieces are combined into the same essential vertex or chain.

If $|c^\ell_1|_{\mathcal{T}_2} \leq 2BCC(\mathcal{T}_1,\mathcal{T}_2)+2$ then there are no safe pieces in $\alpha^\ell_{\Lambda_2}$. Otherwise, decompose the segment $\alpha^\ell_{\Lambda_2}$ as xe_1ye_2z, where

$$|x|_{\mathcal{T}_2} = |z|_{\mathcal{T}_2} = BCC(\mathcal{T}_1,\mathcal{T}_2)$$

and e_1 and e_2 are single edges. Thus any safe essential pieces of $\alpha^\ell_{\Lambda_2}$ is contained in y and the segment e_1ye_2 survives folding (although some of its vertices and edges may be identified).

Consider an essential vertex v in $\alpha^\ell_{\Lambda_2}$. Thus v is adjacent to an $\mathcal{A}_2$–edge of e_1ye_2 not labeled c_2, as well as a $\mathcal{B}_2$–edge (a positively oriented $\{t_0\}$–edge in the case when T_1 is dual to an HNN-extension) of e_1ye_2. Hence, as these edges remain after folding and pruning, v is an essential vertex in $\mathcal{G}^H_2$ unless it is part of a chain. Such a chain could not cross either of the edges of e_1ye_2 that are adjacent to v. Thus such a chain is necessarily essential due to the edges in e_1ye_2 adjacent to v. Similarly, an essential chain in $\alpha^\ell_{\Lambda_2}$ is part of an essential chain in $\mathcal{G}^H_2$ (it may not be maximal in $\mathcal{G}^H_2$).

If $\operatorname{rank}(H) = 1$, then $(\mathcal{G}^H_1)_{\Lambda_2}$ is a circle and as such the segment e_1ye_2 is embedded in $\mathcal{G}^H_2$ and essential vertices and chains of e_1ye_2 are not contained in a larger essential chain of $\mathcal{G}^H_2$. Hence, for $M = 1$, inequality (4.2) holds.

Now suppose that $R \geq \operatorname{rank}(H) > 1$ and v and v' are vertices of $(\mathcal{G}^H_1)_{\Lambda_2}$ that are identified in $\mathcal{G}^H_2$, where v is contained in an essential safe piece arising from $\alpha \in \alpha(\mathcal{G}^H_1)$. There is an edge path β in $(\mathcal{G}^H_1)_{\Lambda_2}$ connecting v to v' which is folded. Notice, the number of edges of β is bounded by $2BCC(\mathcal{T}_1, \mathcal{T}_2)$. As v is not in the extremal $BCC(\mathcal{T}_1, \mathcal{T}_2)$ edges of $\alpha^\ell_{\Lambda_2}$ the path β does not contain a component of $\alpha^\ell_{\Lambda_2} - \{v\}$ and therefore intersects a vertex of valence at least three in $(\mathcal{G}^H_1)_{\Lambda_2}$. For any R, there are boundedly many paths in a graph of rank at most R with at most $2BCC(\mathcal{T}_1, \mathcal{T}_2)$ edges that contain a vertex of valence at least 3. Thus over all chains $\alpha \in \alpha(\mathcal{G}^H_1)$, only boundedly many safe essential pieces of $\alpha^\ell_{\Lambda_2}$ are contained in a given essential vertex of chain of $\mathcal{G}^H_2$ after folding. Taking M to be this bound, inequality (4.2) holds. □

4.4. **Linear growth.** We can now prove our theorem giving a linear lower bound on the free volume of a finitely generated malnormal or cyclic subgroup after iterating a Dehn twist. Although it is not needed here, we also prove the (easier) linear upper bound; this upper bound in applied in [8].

Theorem 4.6. *Let δ_1 be a Dehn twist associated to the very small cyclic tree T_1 with edge stabilizers generated by conjugates of the element c_1 and let T_2 be any other very small cyclic tree. Then there exists a constant $C = C(T_1, T_2)$ such that for any finitely generated malnormal or cyclic subgroup $H \subseteq F_k$ with $\operatorname{rank}(H) \leq R$ and $n \geq 0$ the following hold:*

$$\operatorname{vol}_{T_2}(\delta_1^{\pm n}(H)) \geq \operatorname{vol}_{T_1}(H)\big(n\ell_{T_2}(c_1) - C\big) - M \operatorname{vol}_{T_2}(H) \tag{4.3}$$

$$\operatorname{vol}_{T_2}(\delta_1^{\pm n}(H)) \leq \operatorname{vol}_{T_1}(H)\big(n\ell_{T_2}(c_1) + C\big) + M \operatorname{vol}_{T_2}(H) \tag{4.4}$$

where M is the constant from Proposition 4.5.

Proof. We will only show this for δ_1^n; it will then be clear how to modify the argument for δ_1^{-n}.

Recall that $\mathcal{T}_1 = \mathcal{A}_1 \cup \mathcal{B}_1$ and $\mathcal{T}_2 = \mathcal{A}_2 \cup \mathcal{B}_2$ are bases for F_k relative to the trees T_1 and T_2 respectively, that $\nu\colon \Lambda_1 \to \Lambda_2$ is a homotopy equivalence representing the change in marking and Λ_1 and Λ_2 are the k–petaled roses marked by $\mathcal{T}_1$ and $\mathcal{T}_2$ respectively. Let $B = BCC(\mathcal{T}_1, \mathcal{T}_2)$ denote the bounded cancellation constant with respect to these bases. Finally, let $\rho\colon \mathcal{G}_1^H \to \Lambda_1$ be an immersion of a core graph for H. We will first prove (4.3).

If $\ell_{T_2}(c_1) = 0$ there is nothing to prove. Otherwise, after replacing $\mathcal{T}_1$ by a conjugate (replacing Λ_1 and B accordingly) we can assume that c_1 is T_2–reduced. We can assume that C is large enough so that if $n\ell_{T_2}(c_1) \geq C$ then the segment $\alpha_{\Lambda_2}^{n-1}$ contains a safe essential chain or vertex. Notice that the number of safe essential pieces in $\alpha_{\Lambda_2}^n$ is at least $n\ell_{T_2}(c_1) - (2B+2)$.

Let Υ_n^H be the graph obtained from graph surgery on the core graph $\mathcal{G}_1^H$ along T_1 equipped with the map $\rho_1\colon \Upsilon_n^H \to \Lambda_1$. Notice that at least $\mathrm{vol}_{T_1}(H)$ segments a_n have been added to $\mathcal{G}_1^H$, as every essential piece contains at least one crossing vertex. Further notice that since c_1 is indivisible and cyclically reduced with respect to $\mathcal{T}_1$, the map $\rho_1\colon \Upsilon_n^H \to \Lambda_1$ is an immersion, except possibly at an initial vertex of one of the surgered segments a_n.

By Corollary 4.3 the map $(\nu \circ \rho_1)_{\Lambda_2}\colon (\Upsilon_n^H)_{\Lambda_2} \to \Lambda_2$ folds to an immersion, which by pruning results in the immersion of the core graph $\rho_2^{\delta_1^n(H)}\colon \mathcal{G}_2^{\delta_1^n(H)} \to \Lambda_2$. The image of each of the surgered segments a_n in $(\Upsilon_n^H)_{\Lambda_2}$ is a copy of $\alpha_{\Lambda_2}^n$. We need to bound the number of essential chains and vertices belonging to copies of the segment $\alpha_{\Lambda_2}^n$ in $\Upsilon_{\Lambda_2}^H$ which get pruned. As the order in which folding occurs to arrive at $\mathcal{G}_2^H$ does not matter, we will focus on a single surgered segment a_n and its associated copy of $\alpha_{\Lambda_2}^n$ in $\Upsilon_{\Lambda_2}^H$. Therefore, we assume that the only places where the map $(\Upsilon_n^H)_{\Lambda_2} \to \Lambda_2$ is not an immersion are the initial and terminal vertices of this copy of $\alpha_{\Lambda_2}^n$.

If a_n is surgered in at an essential vertex, then $\rho_1\colon \Upsilon_n^H \to \Lambda_1$ folds at most the first $|c_1|_{\mathcal{T}_1}$ edges of a_n, as any additional folding would imply the presence of an essential maximal chain adjacent to the essential vertex (this uses the fact that c_1 is indivisible in F_k). In particular, the terminal subsegment a_{n-1} survives. Hence after graph composition using ν, at most the extremal B edges of the corresponding copy of $\alpha_{\Lambda_2}^{n-1}$ are pruned. As no other edges of $\mathcal{G}_2^H$ intersect the remaining segment of $\alpha_{\Lambda_2}^{n-1}$ all safe pieces of $\alpha_{\Lambda_2}^n$ are safe pieces of $\mathcal{G}_2^{\delta_1^n(H)}$.

Now suppose the crossing vertex is not an essential vertex. Hence there is an essential chain $\alpha = (\gamma_1, \ldots, \gamma_m)$ such that either the crossing vertex is $\gamma_i(0)$ for some i, or the crossing vertex is $\gamma_m(1)$. Without loss of generality (as we are only measuring the contribution of this essential chain), we can assume that the crossing vertex is rightmost along the chain. If it is $\gamma_m(1)$, then as in the proceeding paragraph the map $\rho_1\colon \Upsilon_n^H \to \Lambda_1$ folds at most the first $|c_1|_{\mathcal{T}_1}$ edges of a_n and all safe pieces of the remaining $\alpha_{\Lambda_2}^{n-1}$ are essential vertices or chains of $\mathcal{G}_2^{\delta_1^n(H)}$.

Otherwise the crossing vertex is $\gamma_i(0)$ for some i. Then $\Upsilon_n^H \to \Lambda_1$ will fold more the the initial $|c_1|_{\mathcal{T}_1}$ at the initial vertex of a_n. Here we claim that at most $2B+2$ safe pieces of $\alpha_{\Lambda_2}^n$ that are folded and pruned are not first identified with a safe essential piece of α.

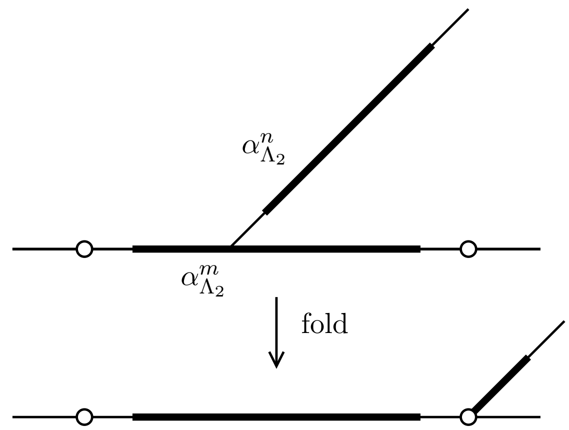

FIGURE 8. Folding the initial part of the surgered segments $\alpha^n_{\Lambda_2}$ to $\Upsilon^H_{\Lambda_2}$. The safe pieces are contained in the thickened edges. At most B more edges of $\alpha^n_{\Lambda_2}$ need to be folded after this initial fold.

If $n \leq m - i$ then in Υ^H_n, the entire segment $\alpha^n_{\Lambda_2}$ can be folded onto $\alpha^m_{\Lambda_2}$, identifying safe pieces of $\alpha^n_{\Lambda_2}$ with safe pieces of $\alpha^m_{\Lambda_2}$; such pieces may then be pruned in forming $\mathcal{G}_2^{\delta_1^n(H)}$. If $n > m - i$, then the terminal $\alpha^{m-i}_{\Lambda_2}$ segment of $\alpha^m_{\Lambda_2}$ can be folded onto $\alpha^n_{\Lambda_2}$. When folding, safe pieces in an initial segment of $\alpha^n_{\Lambda_2}$ are identified with safe pieces of $\alpha^m_{\Lambda_2}$. However some safe pieces of $\alpha^n_{\Lambda_2}$ are identified with non-safe pieces of $\alpha^m_{\Lambda_2}$ coming from essential pieces of $\alpha^m_{\Lambda_2}$ intersecting along the terminal $B+1$ edges of $\alpha^m_{\Lambda_2}$. Thus the number of such safe pieces of $\alpha^n_{\Lambda_2}$ identified with non-safe pieces of $\alpha^m_{\Lambda_2}$ is bounded by $B+1$. There may need to be additional folding at the terminal vertex of $\alpha^m_{\Lambda_2}$; however the amount of folding is bounded. Indeed as α is maximal, at the terminal vertex α in $\mathcal{G}_1^H$, we have to fold at most an additional $|c_1|_{\mathcal{T}_1}$ edges. Thus after folding the initial portion of a_n over α and possibly at most $|c_1|_{\mathcal{T}_1}$ edges, the induced map is an immersion at this vertex and hence at most B of the initial edges in the terminal $\alpha^{n-m-1}_{\Lambda_2}$ segment of $\alpha^n_{\Lambda_2}$ are folded with other edges adjacent to this vertex. Therefore at most an additional B edges are pruned, eliminating at most an additional $B+1$ safe pieces from $\alpha^n_{\Lambda_2}$. This proves our claim. See Figure 8.

Putting this claim together with Proposition 4.5 and summing up over all crossing vertices of $\mathcal{G}_1^H$, we see that the number of essential pieces of $\mathcal{G}_2^{\delta_1^n(H)}$ is bounded below by:

$$\mathrm{vol}_{T_1}(H)\big(n\ell_{T_2}(c_1) - (4B+4)\big) - M(\mathrm{vol}_{T_2}(H) + 1) \tag{4.5}$$

As H is malnormal or cyclic, so is $\delta_1^n(H)$; hence at most one essential chain in $\mathcal{G}_2^{\delta_1^n(H)}$ can be non-simply connected (Lemma 2.3). Thus $\mathrm{vol}_{T_2}(\delta_1^n(H))$ is bounded below by one less than (4.5). Thus for $C = 4B + M + 5$ the inequality (4.3) holds.

We will now sketch the proof for the upper bound (4.4). The idea is similar to the proof of (4.3). Using the machinery developed above, we get an upper bound on the number of essential vertices and chains in $\mathcal{G}_2^{\delta_1^n(H)}$ by looking at essential vertices and chains in $(\Upsilon_n^H)_{\Lambda_2}$. When inserting a copy of a_n at an essential vertex, this may prevent some folding that might have occurred originally in $(\mathcal{G}_1^H)_{\Lambda_2} \to \Lambda_2$, causing some essential vertices and chains in $\mathcal{G}_2^H$ to break into several essential vertices and chains in $\mathcal{G}_2^{\delta_1^n(H)}$. This contribution to $\mathrm{vol}_{T_2}(\delta_1^n(H)$ is controlled by $M \, \mathrm{vol}_{T_1}(H)$. Similar considerations apply at a crossing vertex of an essential chain. At first glance, it may appear that the upper bound in (4.4) is too low as there may be several crossing vertices on a given essential chain. However, it is easy to see that the contribution for all but the rightmost crossing vertex is folded onto to an essential vertex or chain that is already counted. Hence (4.4) holds. □

It is likely that Theorem 4.6 holds for "multi-twists," i.e., products of Dehn twists arising from a single graph of groups decomposition of F_k with cyclic edge stabilizers. This is the case for surfaces, see [21].

Example 4.7. We give an example that shows that the constant C in (4.3) is necessary. Let T_1 be the cyclic tree for the splitting $F_3 = \langle a, c\rangle *_{\langle c\rangle} \langle b, c\rangle$ and $T_2 = T_1\phi$ where ϕ is the outer automorphism of F_3 represented by $a \mapsto b \mapsto c \mapsto ab$; in particular $\ell_{T_2}(c) = 2$. For $g = ac^{-2}bc$ we have $\ell_{T_1}(g) = 2$ and $\phi(g) = a^{-1}b^{-1}a^{-1}cab$ and hence $\ell_{T_2}(g) = 4$. Therefore, if $n = 2$ and $C = 0$, the right hand side of (4.3) is 4. However, $\delta^2(g) = abc^{-1}$ and $\phi(\delta^2(g)) = bcb^{-1}a^{-1}$ and hence $\ell_{T_2}(\delta^2(g)) = 2$. For the two bases $\mathcal{T}_1 = \{a, b, c\}$ and $\mathcal{T}_2 = \{ab, b, c\}$ the bounded cancellation constant $BCC(\mathcal{T}_1, \mathcal{T}_2)$ is 1, and hence, from the proof of Theorem B, we see that we can choose $C = 10$. Upon substituting, the right hand side of (4.3) becomes $4n - 24$. As $\phi(\delta^n(g)) = b(ab)^{n-2}c(ab)^{-(n-1)}$ is reduced for $n \geq 2$, we see that $\ell_{T_2}(\delta^n(g)) = 4n - 6$ for $n \geq 2$.

5. Free factor ping pong

In this section we prove Theorem 5.3, using a variation on the familiar ping pong argument due to Hamidi-Tehrani . As the proof is short, we include it here.

Lemma 5.1 ([19], Lemma 2.4). *Let G be a group generated by g_1 and g_2. Suppose that G acts on a set $\mathcal{X}$, and that there is a function $|\cdot| : \mathcal{X} \to \mathbb{R}_{\geq 0}$ with the following properties: There are mutually disjoint subsets $\mathcal{X}_1$ and $\mathcal{X}_2$ of $\mathcal{X}$ such that if $n > 0$, then $g_i^{\pm n}(\mathcal{X} - \mathcal{X}_i) \subset \mathcal{X}_i$, and for any $x \in \mathcal{X} - \mathcal{X}_i$ we have $|g_i^{\pm n}(x)| > |x|$. Then $G \cong F_2$, and the action on $\mathcal{X}$ of every element $g \in G$ which is not conjugate to a power of some g_i has no periodic points.*

Proof. A non-empty reduced word in g_1 and g_2 is conjugate to a reduced word $w = g_1^{\epsilon_1} \cdots g_1^{\epsilon_2}$, where ϵ_1 and ϵ_2 are non-zero integers. If $x \in \mathcal{X} - \mathcal{X}_1$, then $w(x) \in \mathcal{X}_1$; therefore $w(x) \neq x$ and w is not the identity. If an element of G which is not conjugate to a power of g_1 or g_2 has a periodic point, then some power of it has a fixed point. This power is conjugate to a reduced word of the form $w = g_i^{\epsilon_i} \cdots g_j^{\epsilon_j}$, with $i \neq j$ and ϵ_i, ϵ_j non-zero integers. If $x \in \mathcal{X} - \mathcal{X}_j$, then by assumption $|w(x)| > |x|$. On the other hand, if $x \in \mathcal{X}_j$, then $w^{-1}(x) = g_j^{-\epsilon_j} \cdots g_i^{-\epsilon_i}(x)$ so that $|w^{-1}(x)| > |x|$. Hence w does not have any fixed points and therefore no element of G not conjugate to a power of g_1 or g_2 has a periodic point. □

Let T_1 and T_2 be filling very small cyclic trees with edge stabilizers generated by conjugacy classes of the elements c_1 and c_2, respectively. Also let δ_1 and δ_2 be the associated Dehn twists, $M = M(k-1)$ from Proposition 4.5 and C the larger of the constants $C(T_1, T_2)$ and $C(T_2, T_1)$ from Theorem 4.6. We let $\mathcal{X}$ be the set of conjugacy classes of proper free factors and cyclic subgroups of F_k. Since the trees T_1 and T_2 fill we have:

$$\operatorname{vol}_{T_1}(X) + \operatorname{vol}_{T_2}(X) > 0$$

for any $X \in \mathcal{X}$. Choose an irrational number λ (λ will be end up being close to 1) and define sets:

$$\mathcal{X}_1 = \{X \in \mathcal{X} \mid \operatorname{vol}_{T_1}(X) < \lambda \operatorname{vol}_{T_2}(X)\} \text{ and}$$
$$\mathcal{X}_2 = \{X \in \mathcal{X} \mid \operatorname{vol}_{T_2}(X) < \lambda^{-1} \operatorname{vol}_{T_1}(X)\}.$$

Hence $\mathcal{X}$ is the disjoint union of $\mathcal{X}_1$ and $\mathcal{X}_2$. Finally, we define a function $|\cdot|\colon \mathcal{X} \to \mathbb{R}_{\geq 0}$ by:

$$|X| = \operatorname{vol}_{T_1}(X) + \operatorname{vol}_{T_2}(X)$$

We will now show that for some N and $m, n \geq N$, the group $\langle \delta_1^m, \delta_2^n \rangle$ satisfies Lemma 5.1 with the set $\mathcal{X}$ and function $|\cdot|\colon \mathcal{X} \to \mathbb{R}_{\geq 0}$. The proof is the same as for Lemma 3.1 in [19].

Lemma 5.2. *With the above notation:*

1. $\delta_1^{\pm n}(\mathcal{X}_2) \subset \mathcal{X}_1$ *if* $n\ell_{T_2}(c_1) - C \geq (M+1)\lambda^{-1}$.
2. *If* $n\ell_{T_2}(c_1) - C \geq (M+1)\lambda^{-1}$ *and* $X \in \mathcal{X}_2$, *then* $|\delta_1^{\pm n}(X)| > |X|$.
3. $\delta_2^{\pm n}(\mathcal{X}_1) \subset \mathcal{X}_2$ *if* $n\ell_{T_1}(c_2) - C \geq (M+1)\lambda$.
4. *If* $n\ell_{T_1}(c_2) - C \geq (M+1)\lambda$ *and* $X \in \mathcal{X}_1$, *then* $|\delta_2^{\pm n}(X)| > |X|$.

Proof. If $X \in \mathcal{X}_2$, we have $\operatorname{vol}_{T_2}(X) < \lambda^{-1} \operatorname{vol}_{T_1}(X)$, and $\operatorname{rank}(X) \leq k-1$ and so by Theorem 4.6

$$\begin{aligned}
\operatorname{vol}_{T_2}(\delta_1^{\pm n}(X)) &\geq \operatorname{vol}_{T_1}(X)\big(n\ell_{T_2}(c_1) - C\big) - M \operatorname{vol}_{T_2}(X) \\
&> \operatorname{vol}_{T_1}(X)\big(n\ell_{T_2}(c_1) - C\big) - M\lambda^{-1} \operatorname{vol}_{T_1}(X) \\
&= \operatorname{vol}_{T_1}(X)\big(n\ell_{T_2}(c_1) - C - M\lambda^{-1}\big) \\
&= \operatorname{vol}_{T_1}(\delta_1^{\pm n}(X))\big(n\ell_{T_2}(c_1) - C - M\lambda^{-1}\big) \\
&\geq \lambda^{-1} \operatorname{vol}_{T_1}(\delta_1^{\pm n}(X))
\end{aligned}$$

if $n\ell_{T_2}(c_1) - C \geq (M+1)\lambda^{-1}$, and hence $\delta_1^{\pm n}(X) \in \mathcal{X}_1$. This shows (1), and statement (3) is similar. If $X \in \mathcal{X}_2$, we have $\operatorname{vol}_{T_2}(X) < \lambda^{-1} \operatorname{vol}_{T_1}(X)$, and

$\operatorname{rank}(X) \leq k-1$, so again by Theorem 4.6:

$$\begin{aligned}|\delta_1^{\pm n}(X)| &= \operatorname{vol}_{T_1}(\delta_1^{\pm n}(X)) + \operatorname{vol}_{T_2}(\delta_1^{\pm n}(X)) \\ &\geq \operatorname{vol}_{T_1}(X) + \operatorname{vol}_{T_1}(X)\big(n\ell_{T_2}(c_1) - C\big) - M \operatorname{vol}_{T_2}(X) \\ &> \operatorname{vol}_{T_1}(X)\big(n\ell_{T_2}(c_1) - C + 1\big) - M\lambda^{-1} \operatorname{vol}_{T_1}(X) \\ &= \operatorname{vol}_{T_1}(X)\big(n\ell_{T_2}(c_1) - C + 1 - M\lambda^{-1}\big) \\ &\geq \operatorname{vol}_{T_1}(X)\big(1 + \lambda^{-1}\big) \\ &= \operatorname{vol}_{T_1}(X) + \lambda^{-1} \operatorname{vol}_{T_1}(X) \\ &> \operatorname{vol}_{T_1}(X) + \operatorname{vol}_{T_2}(X) = |X|\end{aligned}$$

if $n\ell_{T_2}(c_1) - C \geq (M+1)\lambda^{-1}$. This shows (2), and statement (4) is similar. □

Equipped with this lemma, we are now ready to prove our main result.

Theorem 5.3. *Let δ_1 and δ_2 be the Dehn twists of F_k for two filling cyclic splittings of F_k. Then there exists $N = N(\delta_1, \delta_2)$ such that for $m, n > N$:*

1. *$\langle \delta_1^m, \delta_2^n \rangle$ is isomorphic to the free group on two generators; and*
2. *if $\phi \in \langle \delta_1^m, \delta_2^n \rangle$ is not conjugate to a power of either δ_1^m or δ_2^n, then ϕ is a hyperbolic fully irreducible element of $\operatorname{Out} F_k$.*

Proof. As mentioned in Remark 3.4, without loss of generality, we can assume that δ_1 and δ_2 are associated to very small cyclic trees for F_k. Using the above set-up and notation, let λ be an irrational number such that $\max\{\lambda, \lambda^{-1}\} \leq 2$. Because λ is irrational, the set $\mathcal{X}$ is equal to the disjoint union $\mathcal{X}_1 \sqcup \mathcal{X}_2$. Let N by large enough such that:

$$N\ell_{T_2}(c_1) - C \geq 2(M+1) \text{ and } N\ell_{T_1}(c_2) - C \geq 2(M+1).$$

Then Lemma 5.2 implies that for $m, n \geq N$, the action of the group $\langle \delta_1^m, \delta_2^n \rangle$ on $\mathcal{X}$ satisfies the hypotheses of Lemma 5.1 with the function $|X| = \operatorname{vol}_{T_1}(X) + \operatorname{vol}_{T_2}(X)$. It follows that $\langle \delta_1^m, \delta_2^n \rangle \simeq F_2$. Further, the Lemma 5.1 implies that if $\phi \in \langle \delta_1^m, \delta_2^n \rangle$ is not conjugate to a power of either δ_1^m or δ_2^n then ϕ acts on $\mathcal{X}$ without periodic orbits. As $\mathcal{X}$ contains all of the conjugacy classes of proper free factors, ϕ is fully irreducible. Likewise, as $\mathcal{X}$ contains all of the conjugacy classes of cyclic subgroups, ϕ is hyperbolic. □

Remark 5.4. By applying the ping pong argument using Lemma 5.2 directly to the word $w = \delta_1^{\epsilon_1}\delta_2^{\kappa_1} \cdots \delta_1^{\epsilon_n}\delta_2^{\kappa_n}$ where $n \geq 2$, and $|\epsilon_i|, |\kappa_i| \geq N$, except possibly $\epsilon_1 = 0$ or $\kappa_n = 0$, we can see that w is nontrivial. Additionally, if both $|\epsilon_1|$ and $|\kappa_n|$ are equal to 0, or if both are at least N, then w is a fully irreducible hyperbolic element of $\operatorname{Out} F_k$.

Remark 5.5. Inspired by Hamidi-Tehrani's approach, Mangahas [28] proved that subgroups of the mapping class group have uniform exponential growth with a uniform bound depending only on the surface and not on the subgroup. It is possible that Theorem 5.3 is a step towards proving Mangahas' theorem for $\mathrm{Out}\, F_k$, although much of the machinery she uses for the mapping class group is still undeveloped in the $\mathrm{Out}\, F_k$ setting.

6. Coarse biLipschitz equivalence

Using the techniques developed in Sections 3 and 4 we can now prove that the sum of the free volumes of a proper free factor for two filling very small cyclic trees is biLipschitz equivalent to the free volume of the free factor for any tree in Outer space. Kapovich and Lustig showed this equivalence for a cyclic subgroup [22].

Theorem 6.1. *Let T_1 and T_2 be two very small cyclic trees for F_k that fill, and let $T \in cv_k$. Then there is a constant K such that for any proper free factor or cyclic subgroup $X \subset F_k$:*

$$\frac{1}{K}\operatorname{vol}_T(X) \leq \operatorname{vol}_{T_1}(X) + \operatorname{vol}_{T_2}(X) \leq K \operatorname{vol}_T(X). \tag{6.1}$$

Proof. First, recall that for any trees T and T' in cv_k, there is a constant K_0 such that for any free factor or cyclic group X

$$\frac{1}{K_0}\operatorname{vol}_T(X) \leq \operatorname{vol}_{T'}(X) \leq K_0 \operatorname{vol}_T(X).$$

Thus to prove (6.1) we can just let T be the tree $\tilde{\Lambda}_1$, where $\Lambda_1 = \Lambda_{\mathcal{T}_1}$ and $\mathcal{T}_1$ is a basis for F_k relative to T_1, metrized such that every edge has length 1. Further consider the tree $\tilde{\Lambda}_2$, where $\Lambda_2 = \Lambda_{\mathcal{T}_2}$, and where $\mathcal{T}_2$ is a basis for F_k relative to T_2, again metrized such that every edge has length 1.

Fix a constant K_1 such that for any free factor or cyclic subgroup X

$$\frac{1}{K_1}\operatorname{vol}_{\tilde{\Lambda}_1}(X) \leq \operatorname{vol}_{\tilde{\Lambda}_2}(X) \leq K_1 \operatorname{vol}_{\tilde{\Lambda}_1}(X).$$

As the rank of X is bounded, each edge in $\tilde{\Lambda}_1^X/X$ and $\tilde{\Lambda}_2^X/X$ can be contained in only boundedly many essential chains independent of X. By Theorems 3.9 and 3.13, for $i = 1, 2$, the free volume $\operatorname{vol}_{T_i}(X)$ is equal to the total number of simply connected chains and essential vertices of $\tilde{\Lambda}_i^X/X$. Thus this is less than some constant D times the total number of edges, $\operatorname{vol}_{\tilde{\Lambda}_i}(X)$, of the graph $\tilde{\Lambda}_i^X/X$. Thus we have $\operatorname{vol}_{T_1}(X) \leq D \operatorname{vol}_{\tilde{\Lambda}_1}(X)$ and $\operatorname{vol}_{T_2}(X) \leq D \operatorname{vol}_{\tilde{\Lambda}_2}(X)$, and hence

$$\begin{aligned}\operatorname{vol}_{T_1}(X) + \operatorname{vol}_{T_2}(X) &\leq D \operatorname{vol}_{\tilde{\Lambda}_1}(X) + D \operatorname{vol}_{\tilde{\Lambda}_2}(X) \\ &\leq D \operatorname{vol}_{\tilde{\Lambda}_1}(X) + DK_1 \operatorname{vol}_{\tilde{\Lambda}_1}(X) \\ &= D(K_1 + 1) \operatorname{vol}_{\tilde{\Lambda}_1}(X)\end{aligned}$$

which shows the right hand inequality of (6.1).

By [22, Theorem 1.4], there exists a constant K' such that for $g \in F_k$:

$$\frac{1}{K'}\ell_{\tilde{\Lambda}_1}(g) \leq \ell_{T_1}(g) + \ell_{T_2}(g) \leq K'\ell_{\tilde{\Lambda}_1}(g).$$

This is (6.1) when $X = \langle g \rangle$.

Otherwise, as X is a proper (noncyclic) free factor, deleting vertices of $\Lambda_1^X = \tilde{\Lambda}_1^X/X$ with valence ≥ 3 results in at most $3k-3$ segments; denote these segments by $S(\Lambda_1^X)$. Then for each such segment $\alpha \in S(\Lambda_1^X)$, there is a subsegment $\alpha' \subseteq \alpha$ such that $|\alpha'|_{\mathcal{T}_1} \geq \frac{1}{2}|\alpha|_{\mathcal{T}_1}$ and α' is cyclically reduced with respect to $\mathcal{T}_1$. Hence:

$$\mathrm{vol}_{\tilde{\Lambda}_1}(X) = \sum_{\alpha \in S(\Lambda_1^X)} |\alpha|_{\mathcal{T}_1} \leq 2 \sum_{\alpha \in S(\Lambda_1^X)} |\alpha'|_{\mathcal{T}_1} = 2 \sum_{\alpha \in S(\Lambda_1^X)} \ell_{\tilde{\Lambda}_1}(\alpha').$$

For each such α', let α'_{Λ_2} be its image under graph composition using the change of marking homotopy equivalence $\nu\colon \Lambda_1 \to \Lambda_2$. We can get a lower bound on $\mathrm{vol}_{T_1}(X) + \mathrm{vol}_{T_2}(X)$ by estimating the sum of the number of essential pieces in the segments α' and the number of essential pieces in the segments α'_{Λ_2} that survive after folding $(\Lambda_1^X)_{\Lambda_2} \to \Lambda_2$. Notice that:

$$\begin{aligned}
\sum_{\alpha \in S(\Lambda_1^X)} \#\text{essential pieces in } \alpha' + \#\text{essential pieces in } \alpha'_{\Lambda_2} \\
&= \sum_{\alpha \in S(\Lambda_1^X)} \ell_{T_1}(\alpha') + \ell_{T_2}(\alpha') \\
&\geq \frac{1}{K'} \sum_{\alpha \in S(\Lambda_1^X)} \ell_{\tilde{\Lambda}_1}(\alpha') \\
&\geq \frac{1}{2K'} \mathrm{vol}_{\tilde{\Lambda}_1}(X)
\end{aligned}$$

Let $B = BCC(\mathcal{T}_1, \mathcal{T}_2)$. As in Section 4, we lose at most the extremal B edges of α'_{Λ_2} whilst folding and pruning $(\Lambda_1^X)_{\Lambda_2} \to \Lambda_2$, thus eliminating at most $2B+2$ essential pieces from α_{Λ_2}. Thus we have

$$\mathrm{vol}_{T_1}(X) + \mathrm{vol}_{T_2}(X) \geq \frac{1}{2K'} \mathrm{vol}_{\tilde{\Lambda}_1}(X) - (2B+2)(3k-3).$$

In other words:

$$\mathrm{vol}_{T_1}(X) + \mathrm{vol}_{T_2}(X)\big(1 + (2B+2)(3k-3)\big) \geq \frac{1}{2K'} \mathrm{vol}_{\tilde{\Lambda}_1}(X)$$

as $\mathrm{vol}_{T_1}(X) + \mathrm{vol}_{T_2}(X) \geq 1$. Choosing $K = \max\{K_1 + 1, 2K'\big(1 + (2B + 2)(3k-3)\big)\}$ completes the proof. □

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