SUBGRAPH ENTROPY

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ABSTRACT. Given $r \geq 3$, we prove that there exists $\lambda > 0$ depending only on r so that if G is a metric graph of rank r with metric entropy 1, then there exists a proper subgraph H of G with metric entropy at least λ. This answers a question of the second two authors together with Rieck. We interpret this as a graph theoretic version of the Bers Lemma from hyperbolic geometry, and explain some connections to the pressure metric on the Culler-Vogtmann Outer Space.

1. INTRODUCTION

For any $g \geq 2$, there is a constant β_g such that for any closed hyperbolic surface X there is a collection of $3g-3$ curves on X that are pairwise non-isotopic where each has length at most β_g. The existence of such a constant β_g was first shown by Bers [2], and the minimal such number β_g is called the *Bers constant*. This constant plays a key role in understanding the topology of moduli space, the geometry of the mapping class group and of hyperbolic 3-manifolds, and the structure of Teichmüller space ([9], [15], [12], [5]). Other important estimates include the work of Buser [6], Buser-Seppälä [7], and Schmutz [18] on the growth of the Bers constant and related functions, such as the maximal length of a systole as a function of the topology of a given hyperbolic surface.

Over the last 40 years, techniques from the study of hyperbolic surfaces have played an influential role in studying the moduli spaces of metric graphs and the outer automorphism group of a free group ([3], [4], [8], [10], [17], [1]). For example, the second two authors and Rieck [1] studied a pair of piecewise Riemannian metrics on the Culler-Vogtmann Outer Space that were inspired by the *thermodynamical interpretation* of the Weil-Petersson metric on Teichmüller space, due to Wolpert, McMullen, and Thurston ([20], [13]). It was Thurston who first defined a Riemannian metric on Teichmüller space by considering a limit of Hessians associated to a sequence of "randomly" chosen closed geodesics; later, Wolpert confirmed that this was a multiple of the Weil-Petersson metric [20], and McMullen helped to develop a more general framework for the so-called *thermodynamic formalism* for making sense of this phenomenon [13]. It is therefore no surprise that the metrics studied in [1] are intimately related to counting closed cycles on objects being parameterized by the Outer space, namely, (marked) metric graphs.

In the spirit or porting over intuition from hyperbolic geometry to the setting of graphs and free groups, the main goal of this paper is to derive an analogue of the Bers constant for metric graphs. Of course, depending on how one defines the normalized Outer Space, the most naive translations of Bers' theorem to the setting of graphs is either trivial, or can not possibly hold. We can see this even in the simplest case of a rose R_n on n petals; indeed, perhaps one is interested in finding a maximal collection of pairwise disjoint (except at the basepoint) cycles on R_n each with length at most some β_n. If the normalized Outer Space is defined to consist of metric graphs of total volume 1, this is trivially true (and one can choose $\beta_n = 1$) and not particularly useful. On the other hand, the metrics on Outer Space mentioned in the previous paragraph require that one chooses a different normalization– one that is suited to considering the growth of the collection of all cycles as a function of length– namely, selecting metric graphs with unit *entropy*, the exponential growth rate of the number of reduced cycles. And since there are roses with unit entropy and arbitrarily large volume, no version of Bers' theorem can hold.

Note that we have already stumbled upon a key difference between the geometry of hyperbolic surfaces and metric graphs: any closed hyperbolic surface *automatically* has unit entropy and has area depending only on g. Since one way of deriving Bers' theorem is to use the Gauss-Bonnet theorem and the bound it produces on total area, one might hope there is a way of reframing the argument in terms of entropy instead of area. Indeed, one can prove the following, *and* that it is equivalent to the existence of the Bers constant β_g (see the Section 2 for a proof that the theorem below is equivalent to Bers' theorem, and for all relevant definitions):

Theorem 1.1. *There is a constant $\eta_g > 0$ such that if X is a closed hyperbolic surface of genus g, there is a proper essential subsurface $Y \subsetneq X$ with entropy at least η_g. Furthermore, the existence of η_g is equivalent to the existence of β_g.*

Our main theorem is the direct analog of the first sentence of Theorem 1.1 in the setting of metric graphs:

Theorem 1.2. *For each $r \geq 3$, there is a constant $C_r > 0$ so that given a graph G with $\operatorname{rk} \pi_1(G) = r \geq 3$ and a length function $\ell \in \mathcal{M}^1(G)$, there is a proper subgraph $G' \subset G$ such that $\mathfrak{h}_{G'}(\ell\big|_{G'}) \geq C_r$.*

In addition to achieving a Bers Lemma like result for metric graphs, we place Theorem 1.2 in conversation with the work Kim-Lim [11], which establishes a relation between the entropy of a graph G' obtained from a graph G by adding an additional edge between a pair of initially non-adjacent vertices. In some sense, our work concerns the opposite direction of starting with a unit entropy graph, deleting an edge, and estimating (from below) the entropy of the resulting graph.

As mentioned above, In [1], the authors study a piecewise Riemannian metric on the outer space of a free group whose definition is inspired by the classical Weil–Petersson metric on the Teichmüller space. We show that, as in the case of the Weil–Petersson metric, this metric is incomplete. Unlike the Weil–Petersson metric and the associated mapping class group action, we determine that so long as the rank is at least 4, the completion admits a global fixed point for the $\operatorname{Out}(F_r)$ action. In fact, we conjectured that the entire outer space has finite diameter, but in attempting to prove this conjecture we ran into the need of a lemma along the lines of Theorem 1.2. We therefore reiterate the following conjecture:

Conjecture 1.3. The entropy metric (see [1]) has finite diameter on the outer space for all ranks $r \geq 4$.

We note that, in Section 4 where we prove Theorem 1.2 for roses, we in fact show that the constant C_r for roses can actually be taken to converge to 1 as $r \to \infty$. We conjecture that this is true for general metric graphs:

Conjecture 1.4. The constant C_r from Theorem 1.2 is universally bounded from below in r, and moreover converges to 1 as $r \to \infty$.

In fact, we conjecture that the most extreme case occurs when G is the rank 3 graph with two vertices and four edges as shown in Figure 1 and $\ell = (\log(3), \log(3), \log(3), \log(3))$. Removing any of the four edges results in a proper subgraph G' with entropy $\log(2)/\log(3) \approx 0.6309\ldots$.

Outline of Paper. In Section 2, we recall the basic properties of entropy that we will need, and we do some computations to obtain some helpful inequalities relating the entropies of various roses and barbells. We also spell out a proof of Theorem 1.1. In Section 3, we define a function encoding the scalar one should multiply against the lengths of all but one edge e of a graph in order to

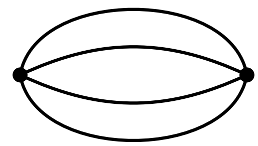

FIGURE 1. The graph G

maintain entropy 1 while blowing up the length of e, and we relate the entropy of the subgraph $G - e$ to the derivative of this function. In Section 4, we prove Theorem 1.2 for roses. In Section 5, we outline a reduction in the argument for the general proof of Theorem 1.2 which will allow us to assume that every non-loop edge of G is incident to a self-loop on both sides. In Section 6, we prove an upper bound on the equilibrium measure of a non-loop edge e in terms of the measures associated to the self-loops incident to either side of e. Finally, in Section 7, we use the bounds in Section 6 complete the proof of Theorem 1.2.

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2. Preliminaries

In this section we collect the necessary definitions and facts needed for the sequel. Much of this material is taken from the works of Aougab–Clay–Rieck [1], Parry–Pollicott [16], and Pollicott–Sharp [17]. We refer to those sources for motivation and further discussion on these topics. In addition, we give a computation in Section 2.3 that will be used in the proof of Theorem 1.2.

2.1. Graphs.

A *graph* is a tuple $G = (V, E, o, \tau, ^{-})$ where:

(1) V and E are sets, called the *vertices* and the *directed edges*,
(2) $o, \tau \colon E \to V$ are functions that specify the *originating* and *terminating* vertices of an edge, and
(3) $^{-}\colon E \to E$ is a fixed-point free involution such that $o(e) = \tau(\bar{e})$.

Given a graph, we fix a subset $E_+ \subset E$ that consists of exactly one edge from each pair $\{e, \overline{e}\}$.

A *length function* on G is a function $\ell\colon E_+ \to \mathbb{R}_{>0}$. The moduli space of all such functions is denoted $\mathcal{M}(G)$. A length function ℓ extends to a function on E by defining $\ell(e) = \ell(\bar{e})$ if $e \notin E_+$.

A (*based*) *circuit* is a finite sequence of edges $(e_1, \ldots, e_n)$ where:

(1) $t(e_i) = o(e_{i+1})$ for $i = 1, \ldots, n-1$ and $t(e_n) = o(e_1)$, and
(2) $e_i \neq \bar{e}_{i+1}$ for $i = 1, \ldots, n-1$ and $e_n \neq \bar{e}_1$.

Given a length function $\ell \in \mathcal{M}(G)$, the length of a circuit $\gamma = (e_1, \ldots, e_n)$ is defined as:

$$\ell(\gamma) = \sum_{i=1}^{n} \ell(e_i).$$

The set of all circuits in G is denoted by $\mathcal{C}(G)$. For a given length function $\ell \in \mathcal{M}(G)$, the subset of circuits whose length is at most t is denoted $\mathcal{C}_{G,\ell}(t)$.

2.2. **Entropy.** The *entropy* of a length function $\ell \in \mathcal{M}(G)$ is defined as:

$$\mathfrak{h}_G(\ell) = \lim_{t\to\infty} \frac{1}{t} \log |\mathcal{C}_{G,\ell}(t)|\,.$$

This function $\mathfrak{h}_G\colon \mathcal{M}(G) \to \mathbb{R}_{\geq 0}$ is real analytic and strictly convex [14, Proposition A.4]. It is readily verified that $\mathfrak{h}_G$ is homogeneous of degree -1. That is:

$$\mathfrak{h}_G(\alpha\ell) = \frac{1}{\alpha}\mathfrak{h}_G(\ell), \tag{2.1}$$

for any $\alpha > 0$ [1, Lemma 3.4].

The subset of length functions with unit entropy is denoted $\mathcal{M}^1(G)$. That is:

$$\mathcal{M}^1(G) = \{\ell \in \mathcal{M}(G) \mid \mathfrak{h}_G(\ell) = 1\}.$$

We associate to the graph G a matrix that records the incident relations. Specifically, we define an $|E| \times |E|$ matrix A_G by

$$A_G(e, e') = \begin{cases} 1 & \text{if } \tau(e) = o(e') \text{ and } \bar{e} \neq e', \\ 0 & \text{else.} \end{cases}$$

Given a length function $\ell \in \mathcal{M}(G)$, we set $A_{G,\ell}$ to be the matrix obtained by multiplying the row of A_G corresponding to $e \in E$ by $\exp(-\ell(e))$ for each edge. That is:

$$A_{G,\ell}(e, e') = \exp(-\ell(e))A_G(e, e').$$

This matrix is irreducible. Moreover, we have that:

$$\mathfrak{h}_G(\ell) = 1 \Leftrightarrow \operatorname{spec}(A_{G,\ell}) = 1 \tag{2.2}$$

where $\operatorname{spec}(\cdot)$ denotes the spectral radius (cf. [17, Lemma 3.1(2)]).

One can use 2.2 to prove easily that a lower bound on entropy implies an upper bound on the length of the shortest cycle:

Lemma 2.1. *Given n and $\delta > 0$, there is ϵ so that if G is a metric graph of rank n and has entropy at least δ, then G admits a cycle γ with $\ell(G) < \epsilon$.*

Proof. If the lemma is false, then there exists a sequence of metric graphs $\{G_n\}_{n\in\mathbb{N}}$ all of the same rank so that $\mathfrak{h}(G_n) \geq \delta$ and so that the length of the shortest cycle on G_n is at least n. Since there are only finitely many graphs of a given rank, we can assume each G_n is of the same topological type; therefore we instead refer to a single graph G with a sequence of metrics $\{\ell_n\}$.

We first prove that there must be an edge, e, with uniformly bounded length over all ℓ_n. If not, then we can assume that every edge has length at least n with respect to ℓ_n. Also, by an application of 2.1 we can assume that $\mathfrak{h}(\ell_n) = 1$. But this is an immediate contradiction, because the entries of A_{G,ℓ_n} converge uniformly to 0, whence it follows that the spectral radii also converge to 0, but 2.2 implies that they must in fact all be 1.

Thus, there is some uniform $\epsilon_1 > 0$ and an edge e so that $\ell_n(e) < \epsilon_1$, for all n. If e is a self-loop, we are done. If not, it follows quite readily from the definitions that the metric graph (G', ℓ'_n) obtained by contracting e to a point and keeping all other edge lengths the same, has entropy at least that of (G, ℓ_n). We can therefore apply the same argument, and induct on the number of edges of G: we eventually arrive at a sequence of edges e_i, each of length at most some ϵ_i (for some ϵ_i depending only on G) for all ℓ_n, which form a cycle on G. It follows that there is some cycle of uniformly bounded length on G, over all ℓ_n, as desired.

∎

We will now use Lemma 2.1 to verify Theorem 1.1 from the introduction, reproduced here for clarity:

Theorem 2.2. *There is a constant $\eta_g > 0$ such that if X is a closed hyperbolic surface of genus g, there is a proper essential subsurface $Y \subsetneq X$ with entropy at least η_g. Furthermore, the existence of η_g is equivalent to the existence of β_g.*

Proof. We first note that the existence of β_g is equivalent to the existence of a single non-boundary-parallel simple closed geodesic of bounded length, which we will denote by β'_g; indeed, assuming that every hyperbolic surface that is not a three-holed sphere admits such a geodesic, we obtain Bers' theorem by applying this assertion iteratively, cutting along the short geodesic whose existence we assume, and inducting on the topology of the surface.

Now, suppose that X admits a proper essential subsurface Y with entropy at least η_g. We claim that without loss of generality, there is some metric graph Γ with a 1-Lipschitz map into Y and a constant $K \geq 1$ (depending only on g) with the property that $\Gamma \hookrightarrow Y$ induces an isomorphism between fundamental groups, and so that if $\alpha \subset Y$ is some closed geodesic, then there is a cycle on Γ homotopic to α and with length at most $K \cdot \ell_X(\alpha)$. Indeed, a simple hyperbolic geometry argument using the Gauss-Bonnet theorem implies that such a Γ and a K exists so that K depends on g and on the diameter of Y, which can not be arbitrarily large if there are no short curves in Y.

The definitions immediately imply that $\mathfrak{h}(\Gamma)$ is uniformly comparable to $\mathfrak{h}(Y)$, and thus by Lemma 2.1, Γ must admit a cycle of length at most some β' which we can take to depend only on g because there are only finitely many topological types of graph that can arise as Γ.

Conversely, assume that any hyperbolic surface X admits a non-boundary-parallel simple closed geodesic α with length at most some β'_g. Then, cutting along α, we obtain a subsurface Y (which may be disconnected– in the event that it is, choose one component at random). Applying the assumption again, we obtain a second simple closed geodesic γ in Y, also with length at most β'_g. Using a similar application of the Gauss-Bonnet theorem as we did above, it follows that either there is a path from α to γ through Y of uniformly bounded length, or, there is some other short simple closed geodesic γ' whose distance to α is less than the distance between γ and α. Applying this argument iteratively yields a metric graph embedded within Y that has the form of a barbell with both self-loops having uniformly bounded length, and with the separating edge *also* having uniformly bounded length in terms only of g. Thus the entropy of this barbell is bounded below in terms of some η_g, and therefore so is that of Y.

□

It follows from 2.2 and the Perron-Frobenius theorem that when $\mathfrak{h}_G(\ell) = 1$, there are positive vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^{2|E|}$ such that:

$$\mathbf{u}^T A_{G,\ell} = \mathbf{u}^T, \quad A_{G,\ell}\mathbf{v} = \mathbf{v}, \text{ and } \mathbf{u}^T\mathbf{v} = 1. \tag{2.3}$$

Moreover, these vectors are unique up to scale. The *equilibrium measure* μ is defined by taking the entry-wise products of $\mathbf{u}$ and $\mathbf{v}$, i.e., $\mu(e) = \mathbf{u}(e)\mathbf{v}(e)$. This determines a measure on the set of bi-infinite lines in G in the usual way, c.f. [19, Section 6.6]. We remark that, by symmetry, one has $\mu(e) = \mu(\bar{e})$.

Notation 2.3. For use later on, we define the following quantities:

$$\begin{aligned} \overline{\mathfrak{h}}_G(\ell) &= \max\left\{\mathfrak{h}_{G'}\left(\ell\big|_{G'}\right) \mid G' \subset G \text{ is a proper subgraph}\right\} \\ \mathfrak{h}_{\inf}(G) &= \inf\left\{\overline{\mathfrak{h}}_G(\ell) \mid \ell \in \mathcal{M}^1(G)\right\} \\ \mathfrak{h}_r &= \inf\left\{\mathfrak{h}_{\inf}(G) \mid \operatorname{rk}\pi_1 G = r\right\} \end{aligned}$$

Theorem 1.2 can be restated as $\mathfrak{h}_r > 0$. As there are only finitely many finite graphs of any given rank without valence one or two vertices, the infimum in the definition of $\mathfrak{h}_r$ is actually a minimum.

We define a function $F_G \colon \mathcal{M}(G) \to \mathbb{R}$ by:

$$F_G(\ell) = \det\left(I - A_{G,\ell}\right). \tag{2.4}$$

From (2.2), it follows that $F_G(\ell) = 0$ if $\mathfrak{h}_G(\ell) = 1$. Furthermore, we have the following relation between F_G and the equilibrium measure μ.

Lemma 2.4. *Let $\ell \in \mathcal{M}^1(G)$ and let μ be the corresponding equilibrium measure. Then $\nabla F(\ell)$ is proportional to μ. That is, there is a nonzero constant C such that $\mu(e) = C\partial_e F(\ell)$ for all $e \in E_+$.*

Proof. This follows as $\mathcal{M}^1(G)$ is a component of $F_G^{-1}(0)$ and also the zero locus of $\log \operatorname{spec}(A_{G,\ell})$ since the gradient of the latter function is μ. See the proofs of Lemmas 3.9 and 4.4 in [1] for complete details. □

Finally, we note that $\mathfrak{h}_G$ extends to a continuous function where we allow ∞ as a value for the length of an edge. Indeed, this follows from the implicit function theorem as entropy is characterized by $\operatorname{spec}(A_{G,\mathfrak{h}_G(\ell)\ell}) = 1$, and since characteristic functions and the Perron–Froebinous eigenvalue depend continuously on the matrix entries.

2.3. **Computations.** We end this section with calculations and bounds on the entropy function that will be needed in the proof of Theorem 1.2.

2.3.1. *The 2–Rose.* We denote the edges of the 2–rose $\mathcal{R}_2$ by e_1 and e_2, and the vertex by v. See Figure 2.

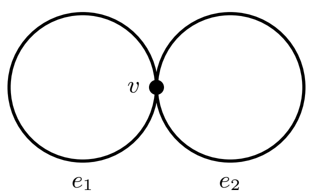

FIGURE 2. The 2–rose graph $\mathcal{R}_2$.

The moduli space $\mathcal{M}^1(\mathcal{R}_2) \subset \mathbb{R}^2_{>0}$ is the zero set of the function:

$$F_{\mathcal{R}_2}(a,b) = 1 - \exp(-a) - \exp(-b) - 3\exp(-a-b).$$

(cf. [17, Section 6.1] or [1, Section 6.1]) In particular, if we define

$$R_2(x) = -\log\left(\frac{1-\exp(-x)}{1+3\exp(-x)}\right) \tag{2.5}$$

then $\mathfrak{h}_{R_2}(x, R_2(x)) = 1$.

2.3.2. *The Barbell.* The graph $\mathcal{B}_2$ has two vertices v and w and three edges e_1, e_2, and e_3 where $o(e_1) = t(e_1) = v$, $o(e_2) = t(e_2) = v$, $o(e_3) = v$, and $t(e_3) = w$. See Figure 3.

We will provide a condition on the length of the edges in $\mathcal{B}_2$ that ensures entropy of at least $\frac{1}{5}$. The following lemma will allow us to compare the entropy of a given barbell to that of a certain rose:

Lemma 2.5. *For any positive constants, a, b, and c, we have that $\mathfrak{h}_{\mathcal{B}_2}(a,b,c) \geq \mathfrak{h}_{\mathcal{R}_2}(a, b+2c)$.*

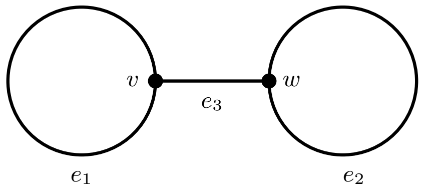

FIGURE 3. The barbell graph $\mathcal{B}_2$.

Proof. Fix a length function $\ell = (a, b, c) \in \mathcal{M}(\mathcal{B}_2)$ and let $\ell' \in \mathcal{M}(\mathcal{R}_2)$ be the length function defined by $\ell' = (a, b + 2c)$.

By identifying the initial and terminal segments of length c of the edge e_2 in $\mathcal{R}_2$, we get a map $\mathcal{R}_2 \to \mathcal{B}_2$ that induces a length decreasing bijection on cycles $\mathcal{C}(\mathcal{R}_2) \to \mathcal{C}(\mathcal{B}_2)$. Hence we find that $\left|\mathcal{C}_{\mathcal{R}_2,\ell'}(t)\right| \leq |\mathcal{C}_{\mathcal{B}_2,\ell}(t)|$ and from this the lemma follows from the definition of entropy. □

Using this comparison, we can now prove a lemma that will be used in the proof of Theorem 1.2 to produce a subgraph of definite entropy.

Lemma 2.6. *Let $\ell \in \mathcal{M}(\mathcal{B}_2)$ be a length function where $\ell(e_1) = 3\exp(-c/2)$, $\ell(e_2) = c/4$ and $\ell(e_3) = c$ for any $c > 0$. Then $\mathfrak{h}_{\mathcal{B}_2}(\ell) \geq \frac{1}{5}$.*

Proof. Fix a positive constant c and let $\ell \in \mathcal{M}(\mathcal{B}_2)$ be as in the statement. Let $a = 3\exp(-c/2)$. Then $\ell(e_2) = -\frac{1}{2}\log(a/3)$ and $\ell(e_3) = -2\log(a/3)$. By Lemma 2.5, we find that:

$$\mathfrak{h}_{\mathcal{B}_2}(\ell) = \mathfrak{h}_{\mathcal{B}_2}(a, -\frac{1}{2}\log(a/3), -2\log(a/3)) \geq \mathfrak{h}_{\mathcal{R}_2}(a, -5\log(a/3)). \tag{2.6}$$

Let $R_2(a) = -\log\left(\frac{1-\exp(-a)}{1+3\exp(-a)}\right)$ as in (2.5). Then $\mathfrak{h}_{\mathcal{R}_2}(a, R_2(a)) = 1$ as explained in Section 2.3.1.

Claim. *For $0 < a < 3$, we have $-5\log(a/3) \leq 5R_2(a)$.*

Proof of Claim. This is equivalent to

$$\exp(a)(1 - a/3) \leq 1 + a. \tag{2.7}$$

The inequality (2.7) is true for $a = 0$. The derivative of the left hand side of (2.7) is $\frac{1}{3}(2-a)\exp(a)$. We observe that

$$\frac{1}{3}(2 - a)\exp(a) \leq 1 \tag{2.8}$$

for $0 \leq a \leq 3$, hence proving the claim, as 1 is the derivative of the right hand side of (2.7). Indeed, the function $\frac{1}{3}(2 - a)\exp(a)$ on $0 \leq a \leq 3$ is maximized at $a = 1$ and we observe that $\frac{1}{3}\exp(1) \leq 1$ holds. □

Therefore, by the claim, we find that:

$$\mathfrak{h}_{\mathcal{R}_2}(a, -5\log(a/3)) \geq \mathfrak{h}_{\mathcal{R}_2}(a, 5R_2(a)) \geq \mathfrak{h}_{\mathcal{R}_2}(5a, 5R_2(a)) = \frac{1}{5}.$$

The lemma now follows by combining this with (2.6). □

3. Computing the Entropy of a Subgraph

Let e be any edge of a graph G and fix a length function $\ell \in \mathcal{M}^1(G)$. We will be interested in a smooth family of length functions $\psi_t \in \mathcal{M}^1(G)$ defined as follows for $0 \leq t < \infty$:

$$\psi_t(e') = \begin{cases} \ell(e) + t & \text{if } e' = e \\ j(t) \cdot \ell(e') & \text{otherwise.} \end{cases} \tag{3.1}$$

where $j\colon \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is defined to be the function so that $\mathfrak{h}_G(\psi_t) = 1$. In other words, we linearly increase the length of e and scale the remaining edges to maintain unit entropy. The smoothness of j follows from the smoothness of the entropy function (for instance, by the implicit function theorem). We call the path $\psi_t \in \mathcal{M}^1(G)$ a *linear time blow up of ℓ along e*.

The main result of this section shows that we can compute the entropy ℓ restricted to the subgraph $G' = G - e$ using the derivative $j'(t)$. Further, we compute a formula for $j'(t)$ that we use in the later sections.

Proposition 3.1. *Let $\ell \in \mathcal{M}^1(G)$ where G is a connected graph with $\operatorname{rk} \pi_1 G \geq 3$, and fix an edge $e \in E(G)$. Let $\psi_t \in \mathcal{M}^1(G)$ be the linear time blow-up of ℓ along e with corresponding scaling function $j\colon \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$, and let μ_t denote the equilibrium measure for the length function ψ_t.*

(1) The derivative of j satisfies:

$$j'(t) = \frac{-\mu_t(e)}{\sum_{e' \neq e} \ell(e')\mu_t(e')}. \tag{3.2}$$

(2) For the subgraph $G' = G - e$ we have:

$$\mathfrak{h}_{G'}(\ell\big|_{G'}) = 1 - \int_0^\infty \left|j'(t)\right| dt. \tag{3.3}$$

Proof. First we show (1). We can compute an expression for $j'(t)$ from the equation $F_G(\psi_t) = 0$. Indeed, differentiating both sides with respect to t yields

$$\sum_{e' \in E(G)} \frac{d}{dt}\psi_t(e') \cdot \partial_{e'} F(\psi_t) = 0.$$

By (3.1) we have that:

$$\frac{d}{dt}\psi_t(e') = \begin{cases} 1 & \text{if } e' = e, \\ j'(t) \cdot \ell(e') & \text{otherwise.} \end{cases}$$

Therefore, we obtain the equation

$$-\partial_e F_G(\psi_t) = \sum_{e' \neq e} j'(t) \cdot \ell(e') \partial_{e'} F_G(\psi_t).$$

Solving for $j'(t)$ yields

$$j'(t) = \frac{-\partial_e F_G(\psi_t)}{\sum_{e' \neq e} \ell(e') \partial_{e'} F_G(\psi_t)}.$$

Finally, the fact that $\nabla F_G(\psi_t)$ is proportional to μ_t, the equilibrium measure at ψ_t, (Lemma 2.4) yields the formula in (3.2).

Next we demonstrate (2). We observe that $j(t)$ is monotonically decreasing and bounded below by 0. This is clear from the definitions as we must shrink the edges in $G' = G - e$ more as the length of e grows, but the explicit formula in (3.2) shows that $j'(t) < 0$ as well. This implies that $\lim_{t\to\infty} j(t)$ exists. Denote this limit by j_∞. We also observe that $j(0) = 1$ as $\mathfrak{h}_G(\ell) = 1$.

Hence, for the right hand side of (3.3), we find:

$$1 - \int_0^\infty |j'(t)|\, dt = 1 + \int_0^\infty j'(t)dt = 1 + \lim_{t\to\infty} j(t) - j(0) = j_\infty.$$

As the rank of G is at least 3, $j_\infty > 0$. Indeed, at least one component of G' has rank at least 2 and hence $\mathfrak{h}_{G'}\left(\ell\big|_{G'}\right) > 0$. This forces $j_t > 1/\mathfrak{h}_{G'}\left(\ell\big|_{G'}\right)$ for all t.

By continuity of the extended entropy function and the definition of ψ_t, we find:

$$1 = \lim_{t\to\infty} \mathfrak{h}_G(\psi_t) = \mathfrak{h}_{G'}(j_\infty \ell\big|_{G'}) = \frac{1}{j_\infty}\mathfrak{h}_{G'}(\ell\big|_{G'}).$$

Thus $\mathfrak{h}_{G'}(\ell\big|_{G'}) = j_\infty$, which verifies (3.3). □

Therefore, our approach for proving Theorem 1.2 will be to bound the improper integral $\int_0^\infty |j'(t)|\, dt$ away from 1.

4. Roses

In this section we will prove Theorem 1.2 for r–roses $\mathcal{R}_r$, $r \geq 3$. As stated in the introduction, our method of proof for the general case requires a non-loop edge. Hence we must deal with the rose case separately. The strategy here though is similar to the general case and provides a good warm-up and overview for the methods used later on. Namely, we will make use of the structure of the matrix $A_{\mathcal{R}_r}$ to get bounds on the values of the equilibrium measure related to edge lengths that, via Proposition 3.1, allow us to estimate the entropy of a subgraph.

These bounds are the context of the following lemma.

Lemma 4.1. *Fix $\ell \in \mathcal{M}^1(\mathcal{R}_r)$ with equilibrium measure μ and fix two edges $e_1, e_2 \in E(\mathcal{R}_r)$. Then:*

$$\exp(\ell(e_1))\mu(e_1) < 4\exp(\ell(e_2))\mu(e_2)$$

Proof. Let $A = A_{\mathcal{R}_r,\ell}$ and fix vectors $\mathbf{u}$ and $\mathbf{v}$ such that $\mathbf{u}^T A = \mathbf{u}^T$, $A\mathbf{v} = \mathbf{v}$, and $\mathbf{u}^T\mathbf{v} = 1$. Then we have that $\mu(e) = \mathbf{u}(e)\mathbf{v}(e)$ for all edges as explained in Section 2.2. Throughout the calculations below we will make use of the fact that $\mathbf{u}(e_i) = \mathbf{u}(\bar{e}_i)$ for any edge e_i and similarly for $\mathbf{v}$ and μ.

As $\mathbf{u}^T A = \mathbf{u}^T$, we find that:

$$\mathbf{u}(e_1) = \exp(-\ell(e_1))\mathbf{u}(e_1) + 2\sum_{i\neq 1}\exp(-\ell(e_i))\mathbf{u}(e_i), \text{and}$$

$$\mathbf{u}(e_2) = \exp(-\ell(e_2))\mathbf{u}(e_2) + 2\sum_{i\neq 2}\exp(-\ell(e_i))\mathbf{u}(e_i).$$

From this we observe that:

$$\mathbf{u}(e_1) < 2\mathbf{u}(e_2). \tag{4.1}$$

As $A\mathbf{v} = \mathbf{v}$, we find that:

$$\mathbf{v}(e_1) = \exp(-\ell(e_1))\left(\mathbf{v}(e_1) + 2\sum_{i\neq 1}\mathbf{v}(e_i)\right), \text{ and}$$

$$\mathbf{v}(e_2) = \exp(-\ell(e_2))\left(\mathbf{v}(e_2) + 2\sum_{i\neq 2}\mathbf{v}(e_i)\right).$$

As above, this implies that:

$$\exp(\ell(e_1))\mathbf{v}(e_1) < 2\exp(\ell(e_2))\mathbf{v}(e_2). \tag{4.2}$$

Combining (4.1) and (4.2) we find:

$$\exp(\ell(e_1))\mu(e_1) = \exp(\ell(e_1))\mathbf{u}(e_1)\mathbf{v}(e_1) < 4\exp(\ell(e_2))\mathbf{u}(e_2)\mathbf{v}(e_2) = 4\exp(\ell(e_2))\mu(e_2). \qquad \square$$

Next, we will show how these bounds prove Theorem 1.2 in the case of roses.

Proposition 4.2. *For each $r \geq 3$, there is a constant $B_r > 0$ so that for any given length function $\ell \in \mathcal{M}^1(\mathcal{R}_r)$, there is a proper subgraph $G' \subset \mathcal{R}_r$ such that $\mathfrak{h}_{G'}\left(\ell\big|_{G'}\right) \geq B_r$. In other words, $\overline{\mathfrak{h}}_{\mathcal{R}_r}(\ell) \geq B_r$ for any length function $\ell \in \mathcal{M}^1(\mathcal{R}_r)$, and hence $\mathfrak{h}_{\inf}(\mathcal{R}_r) \geq B_r$. Moreover,* $\lim_{r\to\infty} B_r = 1$.

Proof. Fix a length function $\ell \in \mathcal{M}^1(\mathcal{R}_r)$ and rename the edges so that $\ell(e_1) \geq \ell(e_2) \geq \ell(e_i)$ for $3 \leq i \leq r$. Let $\psi_t \in \mathcal{M}^1(\mathcal{R}_r)$ be the linear time blow-up of ℓ along e_1 with corresponding scaling function $j(t)$. Let μ_t be the equilibrium measure at ψ_t so that we are in set-up of Proposition 3.1.

By Proposition 3.1(1), we have:

$$\int_0^\infty \left|j'(t)\right|\, dt = \int_0^\infty \frac{\mu_t(e_1)}{\sum_{i\neq 1}\ell(e_i)\mu_t(e_i)}\, dt$$

Lemma 4.1 gives us that $\mu_t(e_1) < 4\exp(j(t)\ell(e_2) - \ell(e_1) - t)\mu_t(e_2)$, and therefore:

$$\int_0^\infty \frac{\mu_t(e_1)}{\sum_{i\neq 1}\mu_t(e_i)\ell(e_i)}\, dt < 4\int_0^\infty \frac{\exp(j(t)\ell(e_2) - \ell(e_1) - t)\mu_t(e_2)}{\sum_{i\neq 1}\ell(e_i)\mu_t(e_i)}\, dt$$

As $j(t)\ell(e_2) \leq \ell(e_2) \leq \ell(e_1)$, we have $\exp(j(t)\ell(e_2) - \ell(e_1)) \leq 1$. This gives:

$$4\int_0^\infty \frac{\exp(j(t)\ell(e_2) - \ell(e_1) - t)\mu_t(e_2)}{\sum_{i\neq 1}\ell(e_i)\mu_t(e_i)}\, dt \leq 4\int_0^\infty \exp(-t)\frac{\mu_t(e_2)}{\sum_{i\neq 1}\ell(e_i)\mu_t(e_i)}\, dt$$

Lastly, since $\ell(e_2)\mu_t(e_2) \leq \sum_{i\neq 1}\ell(e_i)\mu_t(e_i)$, we conclude:

$$\begin{aligned}
\int_0^\infty \left|j'(t)\right|\, dt &\leq 4\int_0^\infty \exp(-t)\frac{\mu_t(e_2)}{\sum_{i\neq 1}\ell(e_i)\mu_t(e_i)}\, dt \\
&= 4\int_0^\infty \frac{\exp(-t)}{\ell(e_2)}\frac{\mu_t(e_2)\ell(e_2)}{\sum_{i\neq 1}\ell(e_i)\mu_t(e_i)}\, dt \\
&\leq \frac{4}{\ell(e_2)}\int_0^\infty \exp(-t)\, dt \\
&= \frac{4}{\ell(e_2)}.
\end{aligned}$$

Let $G' = \mathcal{R}_r - e_1$. Hence by Proposition 3.1(2) we find:

$$\mathfrak{h}_{G'}(\ell\big|_{G'}) = 1 - \int_0^\infty \left|j'(t)\right|\, dt \geq 1 - \frac{4}{\ell(e_2)}. \tag{4.3}$$

Notice that we must have $\ell(e_2) \geq \log(2r-3)$. Indeed, the length function ℓ restricted to the subrose $\mathcal{R}_r - e_1$ has entropy less than 1. For $r \geq 29$, we have $\log(2r-3) > 4$ and therefore, (4.3) gives:

$$\mathfrak{h}_{G'}(\ell\big|_{G'}) \geq 1 - \frac{4}{\ell(e_2)} \geq 1 - \frac{4}{\log(2r-3)} > 0.$$

When $r < 29$, we will need to consider two subcases. First, if $\ell(e_2) > 5$, then (4.3) gives:

$$\mathfrak{h}_{G'}(\ell\big|_{G'}) \geq 1 - \frac{4}{5} = \frac{1}{5}.$$

Next, we suppose that $\ell(e_2) \leq 5$. Thus $\ell(e_i) \leq 5$ for $2 \leq i \leq r$. In this case, we find that:

$$\begin{aligned}\mathfrak{h}_{G'}(\ell\big|_{G'}) &\geq \mathfrak{h}_{\mathcal{R}_{r-1}}(5,\ldots,5)\\ &= \frac{\log(2r-3)}{5}\mathfrak{h}_{\mathcal{R}_{r-1}}(\log(2r-3),\ldots,\log(2r-3))\\ &= \frac{\log(2r-3)}{5}\\ &> \frac{1}{5}.\end{aligned}$$

Hence we can take $B_r = \frac{1}{5}$ for $3 \leq r < 29$, and $B_r = 1 - \frac{4}{\log(2r-3)}$ for $r \geq 29$. □

We conjecture that $B_3 = \log(3)/\log(5) \approx 0.6826...$ which is obtained for the uniform length function $\ell = (\log(5), \log(5), \log(5)) \in \mathcal{M}^1(\mathcal{R}_3)$. Further, we conjecture that $B_r \geq B_3$ for all $r \geq 3$.

5. Reduction

In order to get some control over the combinatorics of the matrix $A_{G,\ell}$ to assist in analyzing the equilibrium measure μ, it is necessary to have some assumptions about the structure of G. In the previous section, we saw that having loop edges allowed for some connections to be made between the components of the vectors $\mathbf{u}$ and $\mathbf{v}$ that ultimately allowed for comparisons on components of μ. In this section we will show how to reduce proving Theorem 1.2 to the case where every vertex has a loop edge. Moreover, we will also show that we can assume the loops are short compared to the non-loop edges.

Definition 5.1. Suppose G is a graph with $\operatorname{rk}\pi_1(G) \leq 3$. We say a sequence of length functions $(\ell_i) \subset \mathcal{M}^1(G)$ is *witnessing* if $\overline{\mathfrak{h}}_G(\ell_i) \to \mathfrak{h}_{\inf}(G)$.

Proposition 5.2. *Suppose $r \geq 3$ and $\mathfrak{h}_r = 0$. Then there is a graph G with $\operatorname{rk}\pi_1(G) = r$, $\mathfrak{h}_{\inf}(G) = 0$, and a witnessing sequence $(\ell_i) \subset \mathcal{M}^1(G)$ such that:*

1. *every vertex of G is incident to a loop edge,*
2. *there exists at least one non-loop edge, and*
3. *for any non-loop edge e, there are loop edges γ_1 and γ_2 with $o(\gamma_1) = t(\gamma_1) = o(e)$ and $o(\gamma_2) = t(\gamma_2) = t(e)$, we have $\ell_i(\gamma_1), \ell_i(\gamma_2) \leq \frac{1}{4}\ell_i(e)$.*

Proof. Fix $r \geq 3$ and suppose that $\mathfrak{h}_r = 0$. As there are only finitely many finite graphs with a fixed finite rank without valence 1 or 2 vertices, we may assume that we have a finite graph G_1 with $\operatorname{rk}\pi_1 G_1 = r$, $\mathfrak{h}_{\inf}(G_1) = 0$ and a witnessing sequence $(\ell_i) \subset \mathcal{M}^1(G_1)$.

First we will show how to arrange (1). Let E_0 be the set of edges of G_1 that are incident to a vertex of G_1 that is not incident to any loop edge. If E_0 is empty, then we set $G' = G_1$. Else, let e_i be the edge in E_0 that minimizes $\ell_i(e_i)$. As there are only finitely many edges, we may assume that the sequence e_i is constant, i.e., $e_i = e$ for some edge $e \in E_0$.

Let G_2 be the graph obtained by collapsing e. There is an induced length function ℓ'_i in G_2 defined by $\ell'_i(e') = \ell_i(e')$ for each remaining edge. Let $\alpha_i = \mathfrak{h}_{G_2}(\ell'_i)$ so that $\hat{\ell}_i = \alpha_i\ell'_i \in \mathcal{M}^1(G_2)$.

Claim 5.3. *For any proper subgraph $H' \subseteq G_2$, there is a proper subgraph $H \subseteq G_1$ such that $\mathfrak{h}_H\left(\ell_i\big|_H\right) \leq \mathfrak{h}_{H'}\left(\ell'_i\big|_{H'}\right) \leq 2\mathfrak{h}_H\left(\ell_i\big|_H\right)$.*

Proof of Claim 5.3. Let H' be a proper subgraph of G_2. There is a unique proper subgraph $H \subset G_1$ that contains e and so that the collapse of e results in H'. There is a bijection between cycles on G_2 and G_1 we denote by: $b\colon \mathcal{C}(G_2) \to \mathcal{C}(G_1)$ which restricts to a bijection between cycles on H' and H.

For any cycle γ on H', by construction, we have that $\ell_i'(\gamma) \leq \ell_i(b(\gamma)) \leq 2\ell_i'(\gamma)$ as any cycle that crossed e necessarily crossed a remaining edge that was longer. Thus

$$|\mathcal{C}_{H,\ell_i}(t)| \leq \left|\mathcal{C}_{H',\ell_i'}(t)\right| \leq |\mathcal{C}_{H,\ell_i}(2t)|\,.$$

The first of these inequalities immediately gives $\mathfrak{h}_H\left(\ell_i\big|_H\right) \leq \mathfrak{h}_{H'}\left(\ell_i'\big|_{H'}\right)$. For the other we compute:

$$\begin{aligned}\mathfrak{h}_{H'}\left(\ell_i'\big|_{H'}\right) &= \lim_{t\to\infty}\frac{1}{t}\log\left|\mathcal{C}_{H',\ell_i'}(t)\right| \\ &\leq \lim_{t\to\infty}\frac{1}{t}\log|\mathcal{C}_{H,\ell_i}(2t)| \\ &\leq \lim_{\tau\to\infty}\frac{2}{\tau}\log|\mathcal{C}_{H,\ell_i}(\tau)| \\ &= 2\mathfrak{h}_H\left(\ell_i\big|_H\right).\end{aligned}$$

□

In particular, we find that $\alpha_i = \mathfrak{h}_{G_2}(\ell_i') \geq \mathfrak{h}_{G_1}(\ell_i) = 1$. Combining this with the other inequality in Claim 5.3 we find:

$$\mathfrak{h}_{H'}\left(\hat{\ell}_i|_{H'}\right) = \frac{1}{\alpha_i}\mathfrak{h}_{H'}\left(\ell_i'|_{H'}\right) \leq \mathfrak{h}_{H'}\left(\ell_i'|_{H'}\right) \leq 2\mathfrak{h}_H\left(\ell_i\big|_H\right).$$

Therefore we have that $\overline{\mathfrak{h}}_{G_2}(\hat{\ell}_i) \leq 2\overline{\mathfrak{h}}_{G_1}(\ell_i)$ and thus $\overline{\mathfrak{h}}_{G_2}(\hat{\ell}_i) \to 0$.

We now repeat this procedure as needed replacing G_1 with G_2 so that the result is a graph G' where every vertex is incident to a loop edge with $\mathfrak{h}_{\text{inf}}(G') = 0$, and a witnessing sequence $(\ell_i) \subset \mathcal{M}^1(G')$. This shows (1)

By Proposition 4.2, $\mathfrak{h}_{\text{inf}}(\mathcal{R}_r) > 0$, hence G' must have a non-loop edge. This shows (2).

Lastly, to show how to arrange for (3), suppose there is a vertex where every incident loop edge has length more than $\frac{1}{4}$ of the length of every incident non-loop edge. Then as we argued in showing (1), we can contract the shortest of these non-loop edges and only change the length of cycles by a bounded amount. An argument similar to the one above then gives the desired result. □

6. Equilibrium Measure Bounds on Non-Loop Edges

The main goal in this section is to prove an analogue of Lemma 4.1 for non-loop edges that arise via the reduction procedure of the previous section.

Lemma 6.1. *Let G be a connected graph and fix a length function $\ell \in \mathcal{M}^1(G)$ with equilibrium measure μ. Suppose that e is a non-loop edge in G that is incident at each of its vertices to a loop edge, denoted γ_1 and γ_2 respectively. Then:*

$$\exp(\ell(e))\mu(e) \leq 2\exp(\ell(\gamma_1) + \ell(\gamma_2))(\mu(\gamma_1) + \mu(\gamma_2))$$

Proof. Fix a length function $\ell \in \mathcal{M}^1(G)$, let $A = A_{G,\ell}$ and fix vectors $\mathbf{u}$ and $\mathbf{v}$ such that $A\mathbf{v} = \mathbf{v}$, $\mathbf{u}^T A = \mathbf{u}^T$ and $\mathbf{u}^T\mathbf{v} = 1$. Then we have that $\mu(e) = \mathbf{u}(e)\mathbf{v}(e)$ as explained in Section 2.2. Denote the edges at $o(e)$ other than e and γ_1 by $e_1, \ldots, e_n$ and denote the edges at $t(e)$ other than $\bar{e}$ and γ_2 by $e_1', \ldots, e_m'$. See Figure 4. Throughout the calculations below we will make use of the fact that $\mathbf{u}(e') = \mathbf{u}(\bar{e}')$ for any edge e' and similarly for $\mathbf{v}$ and μ.

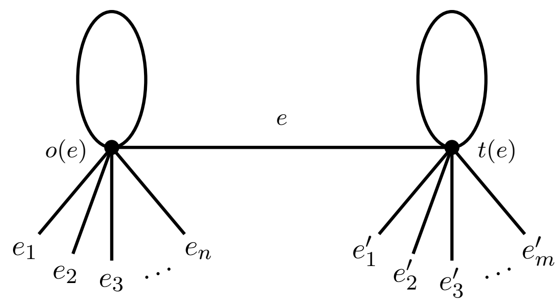

FIGURE 4. The set up in Lemma 6.1.

As $\mathbf{u}^T A = \mathbf{u}^T$, we find that:

$$\mathbf{u}(e) = 2\exp(-\ell(\gamma_1))\mathbf{u}(\gamma_1) + \sum_{i=1}^{n} \exp(-\ell(e_i))\mathbf{u}(e_i). \tag{6.1}$$

We will denote the right-hand side of (6.1) by $X_{\mathbf{u}}$ so that $\mathbf{u}(e) = X_{\mathbf{u}}$. Similarly, as $A\mathbf{v} = \mathbf{v}$, we find that:

$$\mathbf{v}(e) = \exp(-\ell(e)) \left(2\mathbf{v}(\gamma_2) + \sum_{i=1}^{m} \mathbf{v}(e_i') \right). \tag{6.2}$$

We denote $2\mathbf{v}(\gamma_2) + \sum_{i=1}^{n} \mathbf{v}(e_i)$ by $X_{\mathbf{v}}$ so that $\mathbf{v}(e) = \exp(-\ell(e))X_{\mathbf{v}}$. Thus combining (6.1) and (6.2) we have

$$\mu(e) = \mathbf{u}(e)\mathbf{v}(e) = \exp(-\ell(e))X_{\mathbf{u}}X_{\mathbf{v}}. \tag{6.3}$$

We record the same calculations for $\bar{e}$ and find:

$$\mathbf{u}(\bar{e}) = 2\exp(-\ell(\gamma_2))\mathbf{u}(\gamma_2) + \sum_{i=1}^{m} \exp(-\ell(e_i'))\mathbf{u}(e_i') = Y_{\mathbf{u}}$$

$$\mathbf{v}(\bar{e}) = \exp(-\ell(e)) \left(2\mathbf{v}(\gamma_1) + \sum_{i=1}^{n} \mathbf{v}(e_i) \right) = \exp(-\ell(e))Y_{\mathbf{v}}$$

Thus again we have that:

$$\mu(\bar{e}) = \exp(-\ell(e))Y_{\mathbf{u}}Y_{\mathbf{v}}. \tag{6.4}$$

Next, we look at $\mu(\gamma_1)$ and $\mu(\gamma_2)$. We can compute them as above and compare these expressions to those for $\mu(e)$ and $\mu(\bar{e})$. First we look at $\mu(\gamma_1) = \mathbf{u}(\gamma_1)\mathbf{v}(\gamma_1)$ and estimate these quantities.

$$\mathbf{u}(\gamma_1) = \exp(-\ell(\gamma_1))\mathbf{u}(\gamma_1) + \exp(-\ell(e))\mathbf{u}(e) + \sum_{i=1}^{n} \exp(-\ell(e_i))\mathbf{u}(e_i) \geq \frac{X_{\mathbf{u}}}{2} \tag{6.5}$$

$$\mathbf{v}(\gamma_1) = \exp(-\ell(\gamma_1)) \left(\mathbf{v}(\gamma_1) + \mathbf{v}(e) + \sum_{i=1}^{n} \mathbf{v}(e_i) \right) \geq \exp(-\ell(\gamma_1))\frac{Y_{\mathbf{v}}}{2} \tag{6.6}$$

In a similar manner, we also can estimate that $\mathbf{u}(\gamma_2) \geq \frac{Y_{\mathbf{u}}}{2}$ and $\mathbf{v}(\gamma_2) \geq \exp(-\ell(\gamma_2))\frac{X_{\mathbf{v}}}{2}$.

Combining (6.5) and (6.6) together with these related inequalities, we find:

$$\begin{aligned}\mu(\gamma_1)+\mu(\gamma_2) &= \mathbf{u}(\gamma_1)\mathbf{v}(\gamma_1)+\mathbf{u}(\gamma_2)\mathbf{v}(\gamma_2)\\ &\geq \exp(-\ell(\gamma_1))\frac{X_{\mathbf{u}}Y_{\mathbf{v}}}{4}+\exp(-\ell(\gamma_2))\frac{X_{\mathbf{v}}Y_{\mathbf{u}}}{4}\\ &\geq \frac{1}{4}\exp(-\ell(\gamma_1)-\ell(\gamma_2))\left(X_{\mathbf{u}}Y_{\mathbf{v}}+X_{\mathbf{v}}Y_{\mathbf{u}}\right)\end{aligned}$$

As $\mu(e)=\mu(\bar{e})$, we have that $X_{\mathbf{u}}X_{\mathbf{v}}=Y_{\mathbf{u}}Y_{\mathbf{v}}$ by (6.3) and (6.4). Hence $X_{\mathbf{v}}=\frac{Y_{\mathbf{u}}Y_{\mathbf{v}}}{X_{\mathbf{u}}}$ and $Y_{\mathbf{v}}=\frac{X_{\mathbf{u}}X_{\mathbf{v}}}{Y_{\mathbf{u}}}$. Continuing the above inequalities, we find:

$$\begin{aligned}\mu(\gamma_1)+\mu(\gamma_2) &\geq \frac{1}{4}\exp(-\ell(\gamma_1)-\ell(\gamma_2))\left(X_{\mathbf{u}}\left(\frac{X_{\mathbf{u}}X_{\mathbf{v}}}{Y_{\mathbf{u}}}\right)+Y_{\mathbf{u}}\left(\frac{Y_{\mathbf{u}}Y_{\mathbf{v}}}{X_{\mathbf{u}}}\right)\right)\\ &= \frac{1}{4}\exp(-\ell(\gamma_1)-\ell(\gamma_2))\left(X_{\mathbf{u}}X_{\mathbf{v}}\left(\frac{X_{\mathbf{u}}}{Y_{\mathbf{u}}}\right)+Y_{\mathbf{u}}Y_{\mathbf{v}}\left(\frac{Y_{\mathbf{u}}}{X_{\mathbf{u}}}\right)\right)\\ &= \frac{X_{\mathbf{u}}X_{\mathbf{v}}}{4}\exp(-\ell(\gamma_1)-\ell(\gamma_2))\left(\frac{X_{\mathbf{u}}}{Y_{\mathbf{u}}}+\frac{Y_{\mathbf{u}}}{X_{\mathbf{u}}}\right)\\ &\geq \frac{X_{\mathbf{u}}X_{\mathbf{v}}}{2}\exp(-\ell(\gamma_1)-\ell(\gamma_2)).\end{aligned}$$

The last inequality holds as $x+\frac{1}{x}\geq 2$ for $x\in(0,\infty)$. Rewriting, we have:

$$X_{\mathbf{u}}X_{\mathbf{v}}\leq 2\exp(\ell(\gamma_1)+\ell(\gamma_2))(\mu(\gamma_1)+\mu(\gamma_2)) \tag{6.7}$$

The inequality (6.7) together with (6.3) gives:

$$\exp(\ell(e))\mu(e)=X_{\mathbf{u}}X_{\mathbf{v}}\leq 2\exp(\ell(\gamma_1)+\ell(\gamma_2))(\mu(\gamma_1)+\mu(\gamma_2))$$

as desired. □

7. The Proof of Theorem 1.2

In this section, we assemble the proof of Theorem 1.2 and show that $\mathfrak{h}_r>0$. To this end, suppose the theorem is false, that is $\mathfrak{h}_r=0$ for some fixed $r\geq 3$. Let G be the graph with witnessing sequence $(\ell_i)\subset\mathcal{M}^1(G)$ as given by Proposition 5.2. Fix a non-loop edge e in G and loop edges, γ_1,γ_2, incident to the vertices $o(e)$ and $t(e)$ respectively. Recall that by (3) in Proposition 5.2, we can assume that

$$\ell_i(\gamma_j)\leq\frac{1}{4}\ell_i(e),\ \text{for } j=1,2.$$

For each i, let $\psi_t^{(i)}$ be the linear time blow-up of ℓ_i along e and let $j_i\colon\mathbb{R}_{\geq 0}\to\mathbb{R}_{\geq 0}$ be the corresponding scaling function. Denote the equilibrium measure at $\psi_t^{(i)}$ by $\mu_t^{(i)}$. Let $G'=G-e$. By Proposition 3.1 we have that:

$$\mathfrak{h}_{G'}\left(\ell_i\big|_{G'}\right)=1-\int_0^\infty\left|j_i'(t)\right|\,dt=1-\int_0^\infty\frac{\mu_t^{(i)}(e)}{\sum_{e'\neq e}\ell_i(e')\mu_t^{(i)}(e')}\,dt. \tag{7.1}$$

By Lemma 6.1 and the definition of $\psi_t^{(i)}$, we find that:

$$\int_0^\infty \frac{\mu_t^{(i)}(e)}{\sum_{e'\neq e}\ell(e')\mu_t^{(i)}(e')}\,dt \leq \int_0^\infty \frac{2\exp(\psi_t^{(i)}(\gamma_1)+\psi_t^{(i)}(\gamma_2)-\psi_t^{(i)}(e))(\mu_t^{(i)}(\gamma_1)+\mu_t^{(i)}(\gamma_2))}{\sum_{e'\neq e}\ell_i(e')\mu_t^{(i)}(e')}\,dt$$
$$= \int_0^\infty \frac{2\exp(j_i(t)\ell_i(\gamma_1)+j_i(t)\ell_i(\gamma_2)-\ell_i(e)-t)(\mu_t^{(i)}(\gamma_1)+\mu_t^{(i)}(\gamma_2))}{\sum_{e'\neq e}\ell_i(e')\mu_t^{(i)}(e')}\,dt$$

Let $m^{(i)} = \min\{\ell_i(\gamma_1),\ell_i(\gamma_2)\}$. Then the final integral in the above inequality can be rewritten, using the fact that

$$m^{(i)}(\mu_t^{(i)}(\gamma_1)+\mu_t^{(i)}(\gamma_2)) \leq \sum_{e'\neq e}\mu_t^{(i)}(e')\ell_i(e'),$$

as follows:

$$\int_0^\infty \frac{2\exp(j_i(t)\ell_i(\gamma_1)+j_i(t)\ell_i(\gamma_2)-\ell_i(e)-t)(\mu_t^{(i)}(\gamma_1)+\mu_t^{(i)}(\gamma_2))}{\sum_{e'\neq e}\ell_i(e')\mu_t^{(i)}(e')}\,dt$$
$$= 2\int_0^\infty \frac{\exp(j_i(t)\ell_i(\gamma_1)+j_i(t)\ell_i(\gamma_2)-\ell_i(e)-t)}{m^{(i)}}\frac{m^{(i)}(\mu_t^{(i)}(\gamma_1)+\mu^{(i)}(\gamma_2))}{\sum_{e'\neq e}\ell_i(e')\mu_t^{(i)}(e')}\,dt$$
$$\leq 2\int_0^\infty \frac{\exp(j_i(t)\ell_i(\gamma_1)+j_i(t)\ell_i(\gamma_2)-\ell_i(e)-t)}{m^{(i)}}\,dt.$$

As $j_i(t)\ell_i(\gamma_1) \leq \ell_i(\gamma_1) \leq \frac{1}{4}\ell_i(e)$, we have that $\ell_i(e) - j_i(t)\ell_i(\gamma_1) - j_i(t)\ell(\gamma_2) \geq \frac{1}{2}\ell_i(e)$. Therefore, we find:

$$2\int_0^\infty \frac{\exp(j_i(t)\ell_i(\gamma_1)+j_i(t)\ell_i(\gamma_2)-\ell_i(e)-t)}{m^{(i)}}\,dt \leq \frac{2\exp(-\ell_i(e)/2)}{m^{(i)}}\int_0^\infty \exp(-t)\,dt = \frac{2\exp(-\ell_i(e)/2)}{m^{(i)}}$$

This now gives:

$$\mathfrak{h}_{G'}\left(\ell_i\big|_{G'}\right) = 1 - \int_0^\infty \frac{\mu_t^{(i)}(e)}{\sum_{e'\neq e}\ell_i(e')\mu_t^{(i)}(e')}\,dt \geq 1 - \frac{2\exp(-\ell_i(e)/2)}{m^{(i)}}.$$

Since $\mathfrak{h}_{G'}\left(\ell_i\big|_{G'}\right) \to \mathfrak{h}_{\inf}(G) = 0$ as $i \to \infty$ by assumption, we must have that $\frac{2\exp(-\ell_i(e)/2)}{m^{(i)}} \to 1$ as $i \to \infty$. In particular, for large enough i, we find that $m^{(i)} \leq 3\exp(-\ell_i(e)/2)$.

Therefore, the edges e, γ_1, γ_2 form a sub-barbell $\mathcal{B} \subset G$ so that one of γ_1, γ_2 has length at most $3\exp(-\ell_i(e)/2)$ and so that the other self-loop has length at most $\frac{1}{4}\ell_i(e)$. Lemma 2.6 applies and we conclude that for all sufficiently large i we have $\mathfrak{h}_{\mathcal{B}}\left(\ell_i\big|_{\mathcal{B}}\right) \geq \frac{1}{5}$ and hence $\overline{\mathfrak{h}}_G(\ell_i) \geq \frac{1}{5}$ as well. This is a contradiction.

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