WHEN DOES A RIGHT-ANGLED ARTIN GROUP SPLIT OVER $\mathbb{Z}$?

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Abstract. We show that a right-angled Artin group, defined by a graph Γ that has at least three vertices, does not split over an infinite cyclic subgroup if and only if Γ is biconnected. Further, we compute JSJ–decompositions of 1–ended right-angled Artin groups over infinite cyclic subgroups.

1. Introduction

Given a finite simplicial graph Γ, the right-angled Artin group (RAAG) $A(\Gamma)$ is the group with generating set Γ^0, the vertices of Γ, and with relations $[v, w] = 1$ whenever vertices v and w span an edge in Γ. That is:

$$A(\Gamma) = \langle\, \Gamma^0 \mid [v, w] = 1\, \forall v, w \in \Gamma^0 \text{ that span an edge in } \Gamma \,\rangle$$

Right-angled Artin groups, simple to define, are at the focal point of many recent developments in low-dimensional topology and geometric group theory. This is in part due to the richness of their subgroups, in part due to their interpretation as an interpolation between free groups and free abelian groups and also in part due to the frequency at which they arise as subgroups of geometrically defined groups. Recent work of Agol, Wise and Haglund in regards to the Virtual Haken Conjecture show a deep relationship between 3–manifold groups and right-angled Artin groups [1, 10, 11, 14, 15].

One of the results of this paper computes JSJ–decompositions for 1–ended right-angled Artin groups. This decomposition is a special type of graph of groups decomposition over infinite cyclic subgroups, generalizing to the setting of finitely presented groups a tool from the theory of 3–manifolds. So to begin, we are first concerned with understanding when a right-angled Artin group splits over an infinite cyclic subgroup. Recall, a group G *splits* over a subgroup Z if G can be decomposed as an amalgamated free product $G = A *_Z B$ with $A \neq Z \neq B$ or as an HNN-extension $G = A*_Z$.

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Suppose Γ is a finite simplicial graph. A subgraph $\Gamma_1 \subseteq \Gamma$ is *induced* if two vertices of Γ_1 span an edge in Γ_1 whenever they span an edge in Γ. If $\Gamma_1 \subseteq \Gamma$ is a induced subgraph, then the natural map induced by subgraph inclusion $A(\Gamma_1) \to A(\Gamma)$ is injective. A vertex $v \in \Gamma^0$ is a *cut vertex* if the induced subgraph spanned by the vertices $\Gamma^0 - \{v\}$ has more connected components than Γ. A graph Γ is *biconnected* if for each vertex $v \in \Gamma^0$, the induced subgraph spanned by the vertices $\Gamma^0 - \{v\}$ is connected. In other words, Γ is biconnected if Γ is connected and does not contain a cut vertex. Note, K_2, the complete graph on two vertices, is biconnected.

Remark 1.1. There is an obvious sufficient condition for a right-angled Artin group to split over a subgroup isomorphic to $\mathbb{Z}$. (In what follows we will abuse notation and simply say that the group splits over $\mathbb{Z}$.) Namely, if a finite simplicial graph Γ contains two proper induced subgraphs $\Gamma_1, \Gamma_2 \subset \Gamma$ such that $\Gamma_1 \cup \Gamma_2 = \Gamma$ and $\Gamma_1 \cap \Gamma_2 = v \in \Gamma^0$, then $A(\Gamma)$ splits over $\mathbb{Z}$. Indeed, in this case we have $A(\Gamma) = A(\Gamma_1) *_{A(v)} A(\Gamma_2)$.

If Γ has at least three vertices, such subgraphs exist if and only if Γ is disconnected or has a cut vertex, i.e., Γ is not biconnected.

Our first theorem, proved in Section 2, states that this condition is necessary as well.

Theorem A ($\mathbb{Z}$–splittings of RAAGs)**.** *Suppose Γ is a finite simplicial graph that has at least three vertices. Then Γ is biconnected if and only if $A(\Gamma)$ does not split over $\mathbb{Z}$.*

If Γ has one vertex, then $A(\Gamma) \cong \mathbb{Z}$, which does not split over $\mathbb{Z}$. If Γ has two vertices, then $A(\Gamma) \cong F_2$ or $A(\Gamma) \cong \mathbb{Z}^2$, both of which do split over $\mathbb{Z}$ as HNN-extensions.

Remark 1.2. We recall for the reader the characterization of splittings of right-angled Artin groups over the trivial subgroup. Suppose Γ is a finite simpicial graph with at least two vertices. Then Γ is connected if and only if $A(\Gamma)$ is freely indecomposable, equivalently 1–ended. See for instance [4].

In Section 3, for 1–ended right-angled Artin groups $A(\Gamma)$ we describe a certain graph of groups decomposition, $\mathcal{J}(\Gamma)$, with infinite cyclic edge groups. The base graph for $\mathcal{J}(\Gamma)$ is defined by considering the biconnected components of Γ, taking special care with the K_2 components that contain a valence one vertex from the original graph Γ. Our second theorem shows that this decomposition is a JSJ–decomposition.

Theorem B (JSJ–decompositions of RAAGs)**.** *Suppose* Γ *is a connected finite simplicial graph that has at least three vertices. Then* $\mathcal{J}(\Gamma)$ *is a JSJ–decomposition for* $A(\Gamma)$*.*

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2. Splittings of RAAGs over $\mathbb{Z}$

This section contains the proof of Theorem A. The outline is as follows. First, we will exhibit a family of right-angled Artin groups that do not split over $\mathbb{Z}$. Then we will show how if $A(\Gamma)$ is sufficiently covered by subgroups that do not split over $\mathbb{Z}$, then neither does $A(\Gamma)$. Finally, we will show how to find enough subgroups to sufficiently cover $A(\Gamma)$ when Γ has at least three vertices and is biconnected.

Property $\mathbf{F}(\mathcal{H})$. We begin by recalling some basic notions about group actions on trees, see [13] for proofs. In what follows, all trees are simplicial and all actions are without inversions, that is $ge \neq \bar{e}$ for all $g \in G$ and edges e. When a group G acts on a tree T, the *length* of an element $g \in G$ is $|g| = \inf\{d_T(x, gx) \mid x \in T\}$ and the *characteristic subtree* is $T_g = \{x \in T \mid d_T(x, gx) = |g|\}$. The characteristic subtree is always non-empty. If $|g| = 0$, then g is said to be *elliptic* and T_g consists of the set of fixed points. Else, $|g| > 0$ and g is said to be *hyperbolic*, in which case T_g is a linear subtree, called the *axis* of g, and g acts on T_g as a translation by $|g|$.

The following property puts some control over the subgroups that a given group can split over.

Definition 2.1. Suppose $\mathcal{H}$ is a collection of groups. We say a group G *has property* $\mathbf{F}(\mathcal{H})$ if whenever G acts on a tree, then either there is a global fixed point or G has a subgroup isomorphic to some group in $\mathcal{H}$ that fixes an edge.

If $\mathcal{H} = \{H\}$ we will write $\mathbf{F}(H)$.

Remark 2.2. Bass–Serre theory [13] implies that if G has property $\mathbf{F}(\mathcal{H})$ and G splits over a subgroup Z, then Z has a subgroup isomorphic to some group in $\mathcal{H}$.

For the sequel we consider the collection $\mathcal{H} = \{F_2, \mathbb{Z}^2\}$, where F_2 is the free group of rank 2. We can reformulate the question posed in the title using the following proposition.

Proposition 2.3. *Suppose Γ is a finite simplicial graph that has at least three vertices. Then $A(\Gamma)$ has property $\mathbf{F}(\mathcal{H})$ if and only if $A(\Gamma)$ does not split over $\mathbb{Z}$.*

Proof. Bass–Serre theory (Remark 2.2) implies that if $A(\Gamma)$ has property $\mathbf{F}(\mathcal{H})$ then $A(\Gamma)$ does not split over $\mathbb{Z}$.

Conversely, suppose that $A(\Gamma)$ does not split over $\mathbb{Z}$ and $A(\Gamma)$ acts on a tree T without a global fixed point. The stabilizer of any edge is non-trivial as freely decomposable right-angled Artin groups whose defining graphs have at least three vertices split over $\mathbb{Z}$ (Remarks 1.1 and 1.2).

We claim the stabilizer of any edge contains two elements that do not generate a cyclic group. As a subgroup generated by two elements in a right-angled Artin group is either abelian or isomorphic to F_2 [2], this shows that $A(\Gamma)$ has property $\mathbf{F}(\mathcal{H})$. To prove the claim, let Z denote the stabilizer of some edge of T and suppose $\langle g, h \rangle \cong \mathbb{Z}$ for all $g, h \in Z$. Thus Z is abelian. Since abelian subgroups of right-angled Artin groups are finitely generated (as the Salvetti complex is a finite $K(A(\Gamma), 1)$ [5]) we have $Z \cong \mathbb{Z}$. But this contradicts our assumption that $A(\Gamma)$ does not split over $\mathbb{Z}$. $\square$

Thus we are reduced to proving that property $\mathbf{F}(\mathcal{H})$ is equivalent to biconnectivity for right-angled Artin groups whose defining graph has at least three vertices.

A family of right-angled Artin groups that do not split over $\mathbb{Z}$. The following simple lemma of Culler–Vogtmann relates the characteristic subtrees of commuting elements. As the proof is short, we reproduce it here.

Lemma 2.4 (Culler–Vogtmann [6, Lemma 1.1])**.** *Suppose a group G acts on a tree T and let g and h be commuting elements. Then the characteristic subtree of g is invariant under h. In particular, if h is hyperbolic, then the characteristic subtree of g contains T_h.*

Proof. As $h(T_g) = T_{hgh^{-1}}$ if g and h commute then $h(T_g) = T_g$. If h is hyperbolic, then every h–invariant subtree contains T_h. $\square$

Corollary 2.5. *If $\mathbb{Z}^2$ acts on a tree without a global fixed point, then for any basis $\{g, h\}$, one of the elements must act hyperbolically.*

Proof. Suppose that both g and h are elliptic. As $hT_h = T_h$ and $hT_g = T_g$ by Lemma 2.4, the unique segment connecting T_g to T_h is fixed by h and hence contained in T_h. In other words $T_g \cap T_h \neq \emptyset$ and therefore there is a global fixed point. □

Recall that a *Hamiltonian cycle* in a graph is an embedded cycle that visits each vertex exactly once.

Lemma 2.6. *If Γ is a finite simplicial graph with at least three vertices that contains a Hamiltonian cycle, then $A(\Gamma)$ has property $\mathbf{F}(\mathcal{H})$.*

Proof. Enumerate the vertices of Γ cyclically along the Hamiltonian cycle by $v_1, \ldots, v_n$. Notice that $G_i = \langle v_i, v_{i+1} \rangle \cong \mathbb{Z}^2$ for all $1 \leq i \leq n$ where the indices are taken modulo n.

Suppose that $A(\Gamma)$ acts on a tree T without a global fixed point. Further suppose that G_i does not fix an edge, for all $1 \leq i \leq n$.

There are now two cases.

Case I: *Each G_i fixes a point.* The point fixed by G_i is unique as G_i does not fix an edge, denote it p_i. If the points p_i are all the same, then there is a global fixed point, contrary to the hypothesis. Consider the subtree $S \subset T$ spanned by the p_i. Let p be an extremal vertex of S. There is a non-empty proper subset $P \subset \{1, \ldots, n\}$ such that $p = p_i$ if and only if $i \in P$. Let $i_1, j_0 \in P$ be such that the indices $i_0 = i_1 - 1 \mod n$ and $j_1 = j_0 + 1 \mod n$ do not lie in P. See Figure 1. It is possible that $i_1 = j_0$ or $i_0 = j_1$.

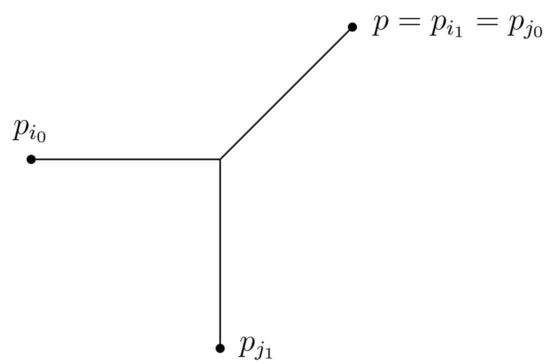

FIGURE 1. A portion of the subtree $S \subset T$ in Case I of Lemma 2.6.

The element $v_{i_1} \in G_{i_0} \cap G_{i_1}$ stabilizes the non-degenerate segment $[p, p_{i_0}]$ and the element $v_{j_1} \in G_{j_0} \cap G_{j_1}$ stabilizes the non-degenerate segment $[p, p_{j_1}]$. As p is extremal, these segments overlap and thus $\langle v_{i_1}, v_{j_1} \rangle$ fixes an edge in T. This subgroup is isomorphic to either F_2 or $\mathbb{Z}^2$.

Case II: *Some G_i does not fix a point.* Without loss of generality, we can assume that G_1 does not fix a point and by Corollary 2.5 that v_2 acts hyperbolically. By Lemma 2.4, v_1 leaves T_{v_2} invariant and so there are integers k_1, k_2, where $k_1 \neq 0$, such that $v_1^{k_1} v_2^{k_2}$ fixes T_{v_2}. Likewise there are integers ℓ_2, ℓ_3, where $\ell_3 \neq 0$ such that $v_2^{\ell_2} v_3^{\ell_3}$ fixes T_{v_2}. Hence $\langle v_1^{k_1} v_2^{k_2}, v_2^{\ell_2} v_3^{\ell_3} \rangle$ fixes T_{v_2}, in particular, this subgroup fixes an edge. This subgroup is isomorphic to either F_2 or $\mathbb{Z}^2$.

In either case, we have found a subgroup isomorphic to either F_2 or $\mathbb{Z}^2$ that fixes an edge. Hence $A(\Gamma)$ has property $\mathbf{F}(\mathcal{H})$. □

Promoting property $\mathbf{F}(\mathcal{H})$. We now show how to promote property $\mathbf{F}(\mathcal{H})$ to $A(\Gamma)$ if enough subgroups have property $\mathbf{F}(\mathcal{H})$.

Proposition 2.7. *Suppose Γ is a connected finite simplicial graph with at least three vertices and suppose that there is a collection $\mathcal{G}$ of induced subgraphs $\Delta \subset \Gamma$ such that:*

(1) *for each $\Delta \in \mathcal{G}$, $A(\Delta)$ has property $\mathbf{F}(\mathcal{H})$, and*
(2) *each two edge segment of Γ is contained in some $\Delta \in \mathcal{G}$.*

Then $A(\Gamma)$ has property $\mathbf{F}(\mathcal{H})$.

Proof. Suppose $A(\Gamma)$ acts on a tree T without a global fixed point.

If for some $\Delta \in \mathcal{G}$, the subgroup $A(\Delta)$ does not have a fixed point, then by (1), $A(\Delta)$, and hence $A(\Gamma)$, contains a subgroup isomorphic to either F_2 or $\mathbb{Z}^2$ that fixes an edge. Therefore, we assume that each $A(\Delta)$ has a fixed point. In particular, each vertex of Γ acts elliptically in T. Also, given three vertices $u, v, w \in \Gamma^0$, such that u and v span an edge as do v and w, the subgroup $\langle u, v, w \rangle$ by (2) is contained in some $A(\Delta)$ and hence has a fixed point. We further may assume the fixed point of such a subgroup $\langle u, v, w \rangle$ to be unique for else $\langle u, v \rangle \cong \mathbb{Z}^2$ fixes an edge.

As there is no global fixed point, there are vertices $v, v' \in \Gamma^0$ that do not share a fixed point. Consider a path from v to v' and enumerate the vertices along this path $v = v_1, \ldots, v_n = v'$. If for some $1 < i < n-1$, the fixed point of $\langle v_{i-1}, v_i, v_{i+1} \rangle$ is different from that of $\langle v_i, v_{i+1}, v_{i+2} \rangle$, then $\langle v_i, v_{i+1} \rangle \cong \mathbb{Z}^2$ fixes an edge as this subgroup stabilizes the non-degenerate segment between the fixed points. If the fixed points are all the same then v and v' have a common fixed point, contrary to our assumptions. □

Proof of Theorem A. Theorem A follows from Proposition 2.3 and the following proposition.

Proposition 2.8. *Suppose Γ is a finite simplicial graph that has at least three vertices. Then Γ is biconnected if and only if $A(\Gamma)$ has property $\mathbf{F}(\mathcal{H})$.*

Proof. Suppose Γ is biconnected. Consider the collection $\mathcal{G}$ of induced subgraphs $\Delta \subseteq \Gamma$ with at least three vertices that contain a Hamiltonian cycle. By Lemma 2.6, each $\Delta \in \mathcal{G}$ has property $\mathbf{F}(\mathcal{H})$.

Consider vertices $u, v, w \in \Gamma^0$ such that u and v span an edge e and v and w span an edge e'. As Γ is biconnected, there is an edge path from u to w that avoids v. Let ρ be the shortest such path and let Δ be the induced subgraph of Γ spanned by v and vertices of ρ. The cycle $e \cup e' \cup \rho$ is a Hamiltonian cycle in Δ and hence $\Delta \in \mathcal{G}$. The two edge segment $e \cup e'$ is contained in Δ by construction.

Hence using the collection $\mathcal{G}$, Proposition 2.7 implies that $A(\Gamma)$ has property $\mathbf{F}(\mathcal{H})$.

Conversely, If Γ is not biconnected, then $A(\Gamma)$ splits over $\mathbb{Z}$ and hence does not have property $\mathbf{F}(\mathcal{H})$ (Remark 1.1 and Proposition 2.3). $\square$

3. JSJ–decompositions of 1–ended RAAGs

We now turn our attention towards understanding all $\mathbb{Z}$–splittings of a 1–ended right-angled Artin group. These are exactly the groups $A(\Gamma)$ with Γ connected and having at least two vertices (Remark 1.2). The technical tool used for understanding splittings over some class of subgroups are *JSJ–decompositions*. There are several loosely equivalent formulations of the notion of a JSJ–decomposition of a finitely presented group, originally defined in this setting and whose existence was shown by Rips–Sela [12]. Alternative accounts and extensions were provided by Dunwoody–Sageev [7], Fujiwara–Papasogalu [8] and Guirardel–Levitt [9].

We have chosen to use Guirardel and Levitt's formulation of a JSJ–decomposition as it avoids many of the technical definitions necessary for the other formulations—most of which have no real significance in the current setting—and as it is particularly easy to verify in the current setting.

In this section we describe a JSJ–decomposition for a 1–ended right-angled Artin group (Theorem B). It is straightforward to verify, given the arguments that follow, that the described graph of groups decomposition is a JSJ–decomposition in the other formulations as well.

JSJ–decompositions à la Guirardel and Levitt. The defining property of a JSJ–decomposition is that it gives a parametrization of all splittings of a finitely presented group G over some special class of subgroups, here the subgroups considered are infinite cyclic. The precise definition is as follows.

Suppose $\mathcal{A}$ is a class of subgroups of G that is closed under taking subgroups and that is invariant under conjugation. An $\mathcal{A}$–*tree* is a tree with an action of G such that every edge stabilizer is in $\mathcal{A}$. An $\mathcal{A}$–tree is *universally elliptic* if its edge stabilizers are elliptic, i.e., have a fixed point, in every $\mathcal{A}$–tree.

Definition 3.1 ([9, Definition 2])**.** A *JSJ–tree* of G over $\mathcal{A}$ is a universally elliptic $\mathcal{A}$–tree T such that if T' is a universally elliptic $\mathcal{A}$–tree then there is a G–equivariant map $T \to T'$, equivalently, every vertex stabilizer of T is elliptic in every universally elliptic $\mathcal{A}$–tree. The associated graph of group decomposition is called a *JSJ–decomposition*.

We will now describe what will be shown to be the JSJ–decomposition of a 1–ended right-angled Artin group.

Suppose Γ is a connected finite simplicial graph with at least three vertices. By B_Γ we denote the *block tree*, that is, the bipartite tree with vertices either corresponding to cut vertices of Γ (black) or bicomponents of Γ, i.e., maximal biconnected induced subgraphs of Γ, (white) with an edge between a black and a white vertex if the corresponding cut vertex belongs to the bicomponent. See Figure 2 for some examples.

For a black vertex $x \in B^0_\Gamma$, denote by v_x the corresponding cut vertex of Γ. For a white vertex $x \in B^0_\Gamma$, denote by Γ_x the corresponding bicomponent of Γ. A white vertex $x \in B^0_\Gamma$ is call *toral* if $\Gamma_x \cong K_2$, the complete graph on two vertices. A toral vertex $x \in B_\Gamma$ that has valence one in B_Γ is called *hanging*.

Associated to Γ and B_Γ is a graph of groups decomposition of $A(\Gamma)$, denoted $\mathcal{J}_0(\Gamma)$. The base graph of $\mathcal{J}_0(\Gamma)$ is obtained from B_Γ by attaching a one-edge loop to each hanging vertex. The vertex group of a black vertex $x \in B^0_\Gamma$ is $G_x = A(v_x) \cong \mathbb{Z}$. The vertex group of a non-hanging white vertex $x \in B_\Gamma$ is $G_x = A(\Gamma_x)$. The vertex group of a hanging vertex $x \in B_\Gamma$ is $G_x = A(v)$ where $v \in \Gamma^0_x$ is the vertex that has valence more than one in Γ. Notice, in this latter case v is a cut vertex of Γ. For an edge $e = [x, y] \subseteq B_\Gamma$ with x black we set $G_e = A(v_x) \cong \mathbb{Z}$ with inclusion maps given by subgraph inclusion. If e is a one-edge loop adjacent to a hanging vertex x, we set $G_e = G_x$ where the two inclusion maps are isomorphisms and the stable letter corresponding to the loop is w where $w \in \Gamma^0_x$ is the vertex that has valence one in Γ.

By collapsing an edge adjacent to each valence two black vertex we obtain a graph of groups decomposition of $A(\Gamma)$, which we denote $\mathcal{J}(\Gamma)$.

It is not necessary for what follows, but we remark that the graph is groups $\mathcal{J}(\Gamma)$ is *reduced* (*in the sense of Bestvina–Feighn* [3]), that is, for each vertex of valence less than three the edge groups are proper subgroups of the vertex group. This property is required for a JSJ–decomposition as defined by Rips–Sela. Observe that all edge groups of $\mathcal{J}(\Gamma)$ are of the form $A(v)$ for some vertex $v \in \Gamma^0$ and in particular maximal infinite cyclic subgroups. By $T_{\mathcal{J}(\Gamma)}$ we denote the associated Bass–Serre tree.

Example 3.2. Examples of B_Γ, $\mathcal{J}_0(\Gamma)$ and $\mathcal{J}(\Gamma)$ for two different graphs are shown in Figure 2. We have $A(\Gamma_1) \cong F_3 \times \mathbb{Z}$. The graph of groups decomposition $\mathcal{J}_0(\Gamma_1)$ is already reduced so $\mathcal{J}(\Gamma_1) = \mathcal{J}_0(\Gamma_1)$. In $\mathcal{J}(\Gamma_1)$ all of the vertex and edge groups are infinite cyclic and all inclusion maps are isomorphisms. Considering the other example, $\mathcal{J}(\Gamma_2)$ corresponds to the graph of groups decomposition $A(\Gamma_2) = \mathbb{Z}^3 *_{\mathbb{Z}} \mathbb{Z}^2 *_{\mathbb{Z}} \mathbb{Z}^3$ where the inclusion maps have image a primitive vector and the images in $\mathbb{Z}^2$ constitute a basis of $\mathbb{Z}^2$.

Proof of Theorem B. Theorem B follows immediately from the following lemma.

Lemma 3.3. *Suppose Γ is a connected finite simplicial graph that has at least three vertices and let $\mathcal{A}$ be the collection of all cyclic subgroups of $A(\Gamma)$. Every vertex stabilizer of $T_{\mathcal{J}(\Gamma)}$ is elliptic in every $\mathcal{A}$–tree.*

In particular, every edge stabilizer of $T_{\mathcal{J}(\Gamma)}$ is elliptic in every $\mathcal{A}$–tree and so $T_{\mathcal{J}(\Gamma)}$ is universally elliptic and every vertex stabilizer of $T_{\mathcal{J}(\Gamma)}$ is elliptic in every universally elliptic $\mathcal{A}$–tree.

Proof. Let T be an $\mathcal{A}$–tree. As $A(\Gamma)$ is 1–ended, every edge stabilizer of T is infinite cyclic. As the vertex groups of a black vertex is a subgroup of the vertex group of some white vertex, we only need to consider white vertices. The vertex group of every non-toral vertex of $\mathcal{J}(\Gamma)$ is elliptic by Proposition 2.8.

Let $x \in B_\Gamma$ be a non-hanging toral vertex. Denote the vertices of $\Gamma_x \cong K_2$ by v_1 and v_2. Then there are vertices $w_1, w_2 \in \Gamma^0$ such that $[v_i, w_j] = 1$ if and only if $i = j$. In other words, the vertices w_1, v_1, v_2, w_2 span an induced subgraph of Γ that is isomorphic to the path graph with three edges.

If $v_1 \in G_x = A(\Gamma_x) \cong \mathbb{Z}^2$ acts hyperbolically, then by Lemma 2.4 the characteristic subtree of both w_1 and v_2 contains T_{v_1}, the axis of v_1. As in the proof of Lemma 2.6, we find integers k_0, k_1, ℓ_0, ℓ_1 with $k_1, \ell_1 \neq 0$ such that $\langle v_1^{k_0} w_1^{k_1}, v_1^{\ell_0} v_2^{\ell_1} \rangle \cong F_2$ fixes T_{v_1} and hence fixes an edge. As every edge stabilizer of T is infinite cyclic, this shows that v_1 must have a fixed point. By symmetry v_2 must also have a fixed point.

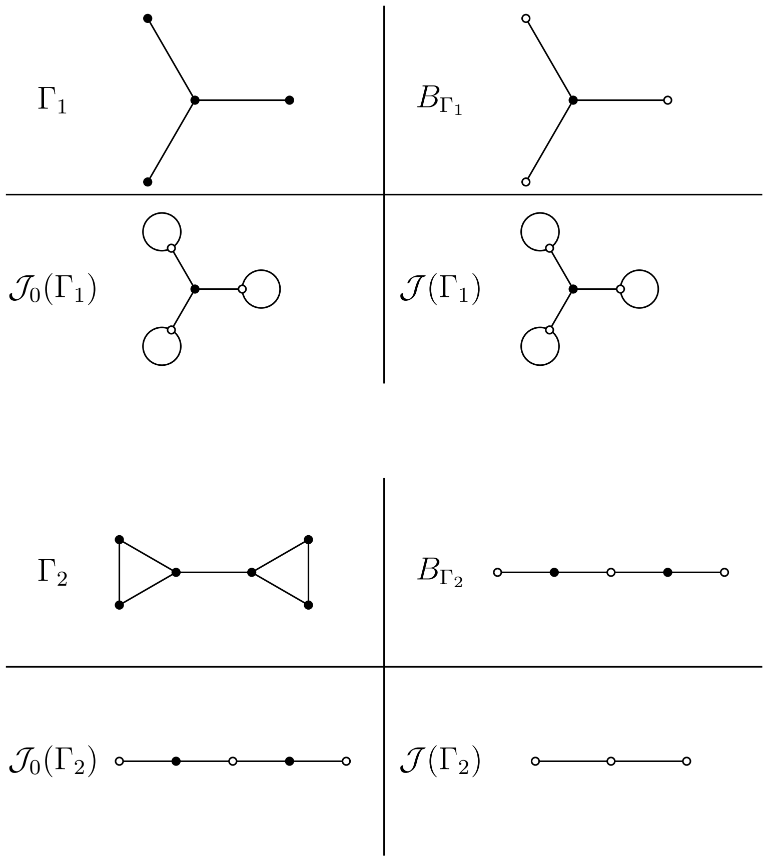

FIGURE 2. Examples of B_Γ, $\mathcal{J}_0(\Gamma)$ and $\mathcal{J}(\Gamma)$.

Since $A(\Gamma_x) = \langle v_1, v_2 \rangle \cong \mathbb{Z}^2$, by Corollary 2.5 this implies that $A(\Gamma_x)$ acts elliptically.

Finally, let $x \in B_\Gamma$ be a hanging vertex. Either G_x is a subgroup of some non-hanging white vertex subgroup and so G_x acts elliptically by the above argument, or $A(\Gamma) \cong F_n \times \mathbb{Z}$ for $n \geq 2$ where G_x is the $\mathbb{Z}$ factor as is the case for Γ_1 in Example 3.2. In the latter case, as G_x is central, by Lemma 2.4 if G_x acts hyperbolically, then $F_n \times \mathbb{Z}$ acts on its axis. Therefore there is a homomorphism $F_n \times \mathbb{Z} \to \mathbb{Z}$ whose kernel fixes an edge. As every edge stabilizer of T is infinite cyclic, G_x must act elliptically. □

We record the following corollary of Lemma 3.3.

Corollary 3.4. *Suppose Γ is a connected finite simplicial graph that has at least three vertices. If $A(\Gamma)$ acts on a tree T such that the stabilizer of every edge is infinite cyclic, then every $v \in \Gamma^0$ that has valence greater than one acts elliptically in T.*

Proof. This follows from Lemma 3.3 as each such vertex is contained in some bicomponent Γ_x for some non-hanging $x \in B_\Gamma$ and hence acts elliptically in $T_{\mathcal{J}(\Gamma)}$. $\square$

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