A FIXED POINT THEOREM FOR DEFORMATION SPACES OF G-TREES

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ABSTRACT. For a finitely generated free group F_n, of rank at least 2, any finite subgroup of $\mathrm{Out}(F_n)$ can be realized as a group of automorphisms of a graph with fundamental group F_n. This result is known as $\mathrm{Out}(F_n)$ realization. This theorem is comparable to Nieslen Realization as proved by Kerckhoff: for a closed surface with negative Euler characteristic, any finite subgroup of the mapping class group can be realized as a group of isometries of a hyperbolic surface. Both of these theorems have restatements in terms of fixed points. For nonnegative integers n we define a class of groups and prove a similar statement for their outer automorphism groups.

For a closed surface with negative Euler characteristic Σ, the mapping class group $\mathcal{MCG}(\Sigma)$ acts on Teichmüller space, the space of hyperbolic metrics on Σ. The stabilizers of this action are finite subgroups of $\mathcal{MCG}(\Sigma)$. Nielsen realization, as proved by Kerckoff [14], states that any finite subgorup of $\mathcal{MCG}(\Sigma)$ can be realized as a group of isometries for a hyperbolic surface. Therefore any finite subgroup of $\mathcal{MCG}(\Sigma)$ fixes a point in T_Σ. In a similar manner, for a finitely generated free group F_n of rank $n \geq 2$, the outer automorphism group $\mathrm{Out}(F_n)$ acts on Outer Space. The stabilizers of this action are finite subgroups of $\mathrm{Out}(F_n)$. Realization for $\mathrm{Out}(F_n)$, as proved by Zimmermann [21], Culler [5] or Khramtsov [15], states that any finite subgroup of $\mathrm{Out}(F_n)$ can be realized as a group of automorphisms of a graph with fundamental group F_n. Thus as for $\mathcal{MCG}(\Sigma)$ and T_Σ, any finite subgroup of $\mathrm{Out}(F_n)$ fixes some point in Outer Space. Both Teichmüller space and Outer Space are contractible [7].

For a nonnegative integer n we introduce a class of groups denoted $\mathfrak{G}(n)$, where the outer automorphism group of any group in this class has a similar realization statement. In other words, for every group $G \in \mathfrak{G}(n)$, there is a contractible space on which $\mathrm{Out}(G)$ acts and we are able to determine that certain subgroups of $\mathrm{Out}(G)$ related to stabilizers have a fixed point. For $n = 0, 1$ we show that any group which is commensurable to a subgroup of a stabilizer actually fixes a point (Corollary 4.2). The class $\mathfrak{G}(0)$ is the class of virtually finitely generated free groups of rank at least 2, thus our result is a generalization of $\mathrm{Out}(F_n)$ realization. In general, we are only able to show that subgroups of $\mathrm{Out}(G)$ commensurable to polycyclic subgroups of stabilizers fix a point.

We define $\mathfrak{G}_0(n)$ as the class of groups which act on a locally finite simplicial tree without an invariant point or line, such that the edge stabilizers are virtually polycyclic subgroups of Hirsch length n. The subset of groups in $\mathfrak{G}_0(n)$ where this action is irreducible and cocompact is denoted $\mathfrak{G}(n)$. In the first section for any finitely generated group G we describe topological spaces $\mathcal{D}$ on

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which certain subgroups of $\mathrm{Out}(G)$ act. These spaces are contractible in most cases. In particular, for $G \in \mathfrak{G}(n)$ we describe a contractible topological space $\mathcal{D}_G$ on which the full group $\mathrm{Out}(G)$ acts. Our main theorem regarding this action is analogous to Nielsen realization for the mapping class group and $\mathrm{Out}(F_n)$ realization.

Main Theorem. *Suppose $G \in \mathfrak{G}(n)$ and W is a polycyclic subgroup of $\mathrm{Out}(G)$ which fixes a point in $\mathcal{D}_G$. If H is a subgroup of $\mathrm{Out}(G)$ commensurable with W, then H fixes a point in $\mathcal{D}_G$.*

The proof of the above is similar to the proof for finite subgroups of $\mathrm{Out}(F_n)$. We review how to prove $\mathrm{Out}(F_n)$ realization. Starting with a finite subgroup W of $\mathrm{Out}(F_n)$, lift this to the subgroup $\widetilde{W}$ in $\mathrm{Aut}(F_n)$. Then $\widetilde{W}$ is virtually free, hence $\widetilde{W}$ acts cocompactly on a simplicial tree T with finite stabilizers by Stallings' theorem [20]. This induces a cocompact free action of $F_n \subseteq \widetilde{W}$ on T. Thus the finite group $W = \widetilde{W}/F_n$ acts on the quotient graph T/F_n, which represents a point in Outer Space. Hence this point is fixed by W.

We seek to mimic this proof. The ingredient we will need is an analog to Stallings' theorem, i.e. when can we raise a splitting of a finite index subgroup to the whole group. For the special case we consider, this question has an answer due to Dunwoody and Roller [8]. We then show that any group which contains a finite index subgroup in $\mathfrak{G}(n)$ is in fact itself in $\mathfrak{G}(n)$. Finally, if $G \in \mathfrak{G}(n)$ and W is a polycyclic subgroup of $\mathrm{Out}(G)$ which stabilizes a point in $\mathcal{D}_G$, we show that $\widetilde{W}$, the lift of W to $\mathrm{Aut}(G)$, is in $\mathfrak{G}(n')$ for some n', inducing an action of G. Thus we can proceed as above for $\mathrm{Out}(F_n)$.

Originally, we were only concerned only with a proof of realization for generalized Baumslag-Solitar groups, the torsion-free groups in $\mathfrak{G}(1)$. However, in doing so it became necessary to prove some statements in greater generality, which provided a proof for any $\mathfrak{G}(n)$-group.

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1. Deformation Spaces of G-trees

For a finitely generated group G, a *G-tree* is a metric simplicial tree on which G acts by isometries. We say two G-trees T and T' are equivalent if there is a G-equivariant isometry between then. When we speak of a G-tree we will always mean the equivalence class of the G-tree. A subgroup is called an *elliptic* subgroup of T if it fixes a point in T. Given a G-tree there are two moves one can perform to the tree that do not change whether or not subgroups of G are elliptic. These moves correspond to the isomorphism $A \cong A *_C C$ and are called *collapse* and *expansion*. For a detailed description of the moves see [10]. In [10] Forester proves the converse, namely if two cocompact G-trees have the same elliptic subgroups, then there is a finite sequence of collapses and expansions (called an *elementary deformation*) transforming one G-tree to the other.

We let $\mathcal{X}$ denote a maximal set of G-trees which are related by an elementary deformation. By the theorem of Forester mentioned above, an equivalent definition is as the set of all G-trees that have the same elliptic subgroups as some fixed G-tree. This set $\mathcal{X}$ is called an *unnormalized deformation space*. We will always assume that the G-trees are minimal and irreducible and that G acts without inversions.

There is an action of $\mathbb{R}^+$ on $\mathcal{X}$ by scaling, the quotient is called a *deformation space* and denoted $\mathcal{D}$. We [4] and independently Guirardel and Levitt [12], [13] have shown that for a finitely generated group, if the actions in $\mathcal{D}$ are irreducible and there is a reduced G-tree with finitely generated vertex stabilizers, then $\mathcal{D}$ is contractible. The topology for the preceding statement is the axes topology induced from the embedding $\mathcal{D} \to \mathbb{RP}^{\mathcal{C}}$ where $\mathcal{C}$ is the set of all conjugacy classes of elements in G, or equivalently the Gromov-Hausdorff topology. See [4] for details.

In general, the space $\mathcal{D}$ is acted on only by a subgroup of $\mathrm{Out}(G)$, where the action is precomposition. This subgroup is the subgroup of $\mathrm{Out}(G)$ which permutes the set of elliptic subgroups associated to $\mathcal{D}$.

If $G \in \mathfrak{G}(n)$ then there is a locally finite G-tree T where all of the stabilizers are virtually polycyclic subgroups of Hirsch length n. We will show in the next section (Lemma 2.1) that the set of elliptic subgroups for this action is invariant under all automorphisms of G. Hence the deformation space containing T is invariant under $\mathrm{Out}(G)$. We denote this space as $\mathcal{D}_G$.

2. Virtually Polycyclic Groups and the Class of $\mathfrak{G}(n)$-Groups

A group G which admits a filtration $\{1\} = G_0 \lhd G_1 \lhd \ldots \lhd G_n = G$ with $G_i \lhd G_{i-1}$ normal and G_{i-1}/G_i cyclic is called *polycyclic*. The *Hirsch length*, denoted $\mathrm{h}\, G$, of a polycyclic group G is the number of infinite cyclic factors is an invariant of G. If G is polycyclic and H is any finite index subgroup then $\mathrm{h}\, H = \mathrm{h}\, G$. In fact, if H is a subgroup of G, then $\mathrm{h}\, H \leq \mathrm{h}\, G$ with equality if and only if H has finite index in G. This allows us to define the Hirsch length of a group which contains a polycyclic group as a finite index subgroup. Such groups are called *virtually polycyclic*. These groups are also referred to as polycyclic-by-finite groups in the literature. Note that if $1 \to K \to G \to H \to 1$ is a short exact sequence then H and K are virtually polycyclic if and only if G is. In this case, the Hirsch lengths satisfy $\mathrm{h}\, G = \mathrm{h}\, H + \mathrm{h}\, K$.

As mentioned in the introduction, originally the main theorem was only intended for generalized Baumslag-Solitar groups. A group G is a *generalized Baumslag-Solitar group* if there is a cocompact G-tree where the stabilizer of any point is isomorphic to $\mathbb{Z}$. As the only $\mathbb{Z}$ subgroups of $\mathbb{Z}$ are finite index, this G-tree must necessarily be locally finite. Hence if G is not isomorphic to $\mathbb{Z}$,$\mathbb{Z} \oplus \mathbb{Z}$ or the Klein-bottle group, then $G \in \mathfrak{G}(1)$. Equivalently, G is a generalized Baumslag-Solitar group if it admits a graph of groups decomposition where all of the edge groups and vertex groups are isomorphic to $\mathbb{Z}$.

Such groups were first studied by Kropholler [16], where it is shown that generalized Baumslag-Solitar groups are the only finitely generated groups of cohomological dimension two that contain an infinite cyclic subgroup which intersects each of its conjugates in a finite index subgroup. It is clear that for a generalized Baumslag-Solitar group any vertex group in the above mentioned graph of groups decomposition satisfies this condition. Forester's Lemma 2.5 in [11] (a generalization of which appears as Lemma 2.1 below) implies that when the action does not have an invariant line, the elliptic subgroups are the only subgroups which satisfy this condition. As this condition is algebraic, the set of elliptic subgroups is invariant under all automorphisms of G, hence we can talk about an $\mathrm{Out}(G)$-invariant deformation space. We now generalize this fact to any $\mathfrak{G}(n)$-group.

Recall that a group G is called *slender* if every subgroup is finitely generated. Virtually polycyclic groups are slender. Slenderness of a group G is equivalent to every subgroup $H \subseteq G$ having property AR: whenever H acts on a simplicial tree, H either stabilizes a point or has an axis [9]. Throughout the following, we use the notation $H^g = gHg^{-1}$. We say two subgroups H, H' of G are *commensurable* if $H \cap H'$ has finite index in both H and H'. The commensurator of a subgroup $H \subseteq G$ is $\mathrm{Comm}_G(H) = \{g \in G \mid H \text{ is commensurable with } H^g\}$.

Lemma 2.1. (Forester [11]) *Let T be locally finite G-tree such that the stabilizer of any point in T is slender. If T does not contain a G-invariant line, then a subgroup $H \subseteq G$ is elliptic if and only if H is contained in a subgroup K, where K is slender and* $\mathrm{Comm}_G(K) = G$.

Proof. As T is a locally finite simplicial tree, any vertex stabilizer is commensurable to all of its conjugates. Hence, if H is elliptic, it is contained in a vertex stabilizer K which satisfies the conclusion of the lemma.

For the converse suppose $H \subseteq G$ does not act elliptically and H is contained in a slender subgroup K. Hence K does not act elliptically. L_K be its axis. Then the axis of K^g is gL_K. If K and K^g are commensurable, then they have the same axis so $L_K = gL_K$. Hence if K is commensurable to all of its conjugates, then L_K is a G-invariant line. □

Thus for such actions the elliptic subgroups are determined algebraically. In particular, the elliptic subgroups for these actions are invariant under $\mathrm{Aut}(G)$. When the action is cocompact, we can talk about an $\mathrm{Out}(G)$-invariant deformation space, denote this space $\mathcal{D}_G$. Hence every point in $\mathcal{D}_G$ is a locally finite G-tree where the stabilizers are virtually polycyclic of Hirsch length n. For $G \in \mathfrak{G}(n)$, as these actions are irreducible, G contains a free subgroup of rank 2. Thus if G acts on a tree T with virtually polycyclic stabilizers of Hirsch length n, then T cannot be a line. This will be used without further mention. We have another lemma due to Forester about the splittings of $\mathfrak{G}(n)$-groups as amalgams over virtually polycyclic groups K with $\operatorname{h} K = n$. Say that G *splits over* K if G can either be written as a nontrivial free product with amalgamation $G = A *_K B$ or as an HNN-extension $G = A*_K$.

Lemma 2.2. (Forester [11]) *Suppose $G \in \mathfrak{G}(n)$ and $T \in \mathcal{D}_G$. If G splits over a virtually polycyclic subgroup K with $\operatorname{h} K = n$, then K fixes a point in T. Moreover, the vertex group(s) in this splitting are $\mathfrak{G}_0(n)$-groups or virtually polycyclic of Hirsch length n or $n+1$ and finitely presented.*

Proof. Let Y be the Bass-Serre tree for the splitting of G over K and H a vertex stabilizer for the G-tree T. Then similarly to Lemma 2.1, H must act elliptically on Y as Y cannot contain a G-invariant line. Let $y \in Y$ be a vertex fixed by H and e an edge stabilized by K. There is some $g \in G$ such that e separates y from gy. As H and H^g are commensurable, there is a finite index subgroup $H' \subseteq H$ stabilizing e, hence contained in K. As both K and H' have Hirsch length n, H' has finite index in K, hence K an H are commensurable. Thus as H fixes a point in any G-tree $T \in \mathcal{D}_G$, so does K.

As for the moreover, suppose A is a vertex group for the splitting of G over K. We examine how A acts on T. If A fixes a point, then A is virtually polycyclic with $\operatorname{h} A = n$. If A has an invariant line on which it acts nontrivially, then there is a short exact sequence $1 \to K' \to A \to \mathbb{Z} \to 1$ or $1 \to K' \to A \to \mathbb{Z}_2 * \mathbb{Z}_2 \to 1$ where K' is commensurable to K. Hence A is virtually polycyclic and $\operatorname{h} A = n+1$. Otherwise this action implies that $A \in \mathfrak{G}_0(n)$.

To see that A is finitely presented we can assume that $A \in \mathfrak{G}_0(n)$. As K acts elliptically in T, using the action of A on T we can refine the splitting of G over K to get a graph of groups decomposition for G that includes the graph of groups decomposition of A with virtually polycyclic vertex and edge groups of Hirsch length n. As G is finitely generated, after reducing we can assume that the graph of groups decomposition for G, hence the graph of groups decomposition of A, is a finite graph. Thus A can be expressed as a finite graph of groups where all of the vertex and edge groups are virtually polycyclic of Hirsch length n. In particular, A is finitely presented. □

We record some properties about $\mathfrak{G}(n)$-groups that will be used in section 4.

Lemma 2.3. *Let $G \in \mathfrak{G}(n)$ then:*

1. $\operatorname{cd}_{\mathbb{Q}} G = n+1$*;*
2. *G does not split over a virtually polycyclic group of Hirsch length less than n; and*
3. *the center of G, $Z(G)$, is a virtually polycyclic subgroup with $\operatorname{h} Z(G) \leq n$. The quotient $G/Z(G)$ is in $\mathfrak{G}(n')$ for $n' = n - \operatorname{h} Z(G)$.*

Proof. For 1. and 2. see Bieri sections 6 and 7 [3].

To see 3., let $T \in \mathcal{D}_G$. As the action is irreducible, $Z(G)$ must act trivially on T [1]. Hence $Z(G)$ is a virtually polycyclic subgroup with $\operatorname{h} Z(G) \leq n$. Also, we have an induced action of $G/Z(G)$ on

T, irreducible and cocompact, and the stabilizers are the quotients of the stabilizers for the G-action by $Z(G)$. Hence $G/Z(G)$ is in $\mathfrak{G}(n')$ where $n' = n - \mathrm{h}\, Z(G)$. $\square$

Remark 2.4. As cohomological dimension is an invariant of the group, if $G \in \mathfrak{G}(n)$, then $G \notin \mathfrak{G}(n')$ for $n \neq n'$. Also note that 2. implies that if $n > 0$ then G is one-ended.

3. Promoting Finite Index Splittings

The main step in proving $\mathrm{Out}(F_n)$ realization is to use Stallings' theorem to get a splitting of the virtually free subgroup of $\mathrm{Aut}(F_n)$ which is the lift of some finite subgroup in $\mathrm{Out}(F_n)$. In the present setting we will need an analog of Stallings' theorem to tell us when a splitting of a finite index subgroup $H \subseteq G$ over $K \subseteq H$ implies a splitting of the whole group G over some subgroup K' which is commensurable to K. In general, we cannot expect a splitting of G. In our special case though, the answer is given by the following theorem of Dunwoody and Roller [8] as stated by Scott and Swarup [19]. A subgroup $H \subseteq G$ has a *large commensurator* if H has infinite index in $\mathrm{Comm}_G(H)$. The ends of the pair of groups $K \subseteq G$ is the number of ends of Γ/K where Γ is a Cayley graph for G. If G splits over a subgroup K, then $e(G, K) \geq 2$. See [19] or [20] for these notions.

Theorem 3.1. (Dunwoody-Roller [8]) *Let G be a one-ended, finitely generated group which does not split over a virtually polycyclic subgroup of Hirsch length less than n, and let K be a virtually polycyclic subgroup of Hirsch length nwith large commensurator, such that $e(G, K) \geq 2$. Then G splits as an amalgam over some subgroup commensurable with K.*

We can now state our analog to Stallings' Theorem.

Theorem 3.2. *Let G be a finitely presented group which has a finite index subgroup $H \in \mathfrak{G}(n)$, then $G \in \mathfrak{G}(n)$.*

Proof. If $n = 0$ then this is Stallings' theorem [20], so we assume that $n \geq 1$.

Let $T \in \mathcal{D}_H$ and K be an edge stabilizer of T. As finite index subgroups of H are in $\mathfrak{G}(n)$, we can assume that H is normal in G. By Lemma 2.1, elliptic subgroups of H are invariant under automorphisms of H, hence $\mathrm{Comm}_G(K) = G$. As $e(H, K) \geq 2$ and H is a finite index subgroup of G, we must have that $e(G, K) \geq 2$. Then by Theorem 3.1, G splits over a subgroup K' commensurable with K. Let T' be the Bass-Serre tree for this splitting of G as an amalgam over K'. If T' is locally finite then we are done as the vertex and edge stabilizers for G acting on this tree are then commensurable to K hence virtually polycyclic subgroups of Hirsch length n, hence $G \in \mathfrak{G}(n)$.

Suppose that T' is not locally finite, we now show that we can split a vertex group for the graph of group decomposition induced by H acting on T'. As T' is not locally finite there is a vertex group H_v which is not a virtually polycyclic subgroup of Hirsch length at most n. Suppose G_v

is the vertex group under the G-action, thus H_v has finite index in G_v. As the induced action of H_v on T is nontrivial, we get a graph of groups decomposition for H_v. Then we can collapse this graph of groups decomposition to get a splitting of H_v over some edge stabilizer K_v. Denote the Bass-Serre tree for this splitting as T_v. As K_v is commensurable to K we have $\mathrm{Comm}_G(K_v) = G$, thus $\mathrm{Comm}_{G_v}(K_v) = G_v$. As H_v is finitely generated, G_v is also. Therefore by as above by Theorem 3.1, G_v splits as an amalgam over K'_v which is commensurable to K_v hence also K'.

As K'_v and K' are commensurable, in the Bass-Serre tree associated to the splitting of G_v over K'_v, K' acts elliptically. This allows us to refine the one edge splitting of G over K' to get a two edge splitting of G over K' and K'_v. Once again, we have a Bass-Serre tree T_0 associated to this graph of groups decomposition, and the action of H on T_0 induces a graph of groups decomposition for H.

If T_0 is not locally finite, repeat. As long as the resulting Bass-Serre tree is not locally finite we can continue. Since at each step, we add one edge to the quotient graph of groups decomposition of G, this process must terminate by Bestvina-Feighn [2]. □

We also note that recently Kropholler has proved a more general statement [17].

4. Realization

In this section we prove the main theorem. For the remainder of this paper, we let $G \in \mathfrak{G}(n)$ be fixed and $\mathcal{D}_G$ the $\mathrm{Out}(G)$-invariant deformation space discussed in section 2.

Suppose that W is a subgroup of $\mathrm{Out}(G)$ and W fixes some point $T \in \mathcal{D}_G$. Then $\widetilde{W}$ the lift of W to $\mathrm{Aut}(G)$ consists of automorphisms ϕ such that there exists an isometry $h_\phi : T \to T$ where $h_\phi(gx) = \phi(g)h_\phi(x)$ for all $x \in T, g \in G$. As the G-trees in $\mathcal{D}_G$ are irreducible and minimal, h_ϕ is unique [6]. Thus we get a homomorphism $\widetilde{W} \to \mathrm{Isom}(T)$, i.e. T is a $\widetilde{W}$-tree. It is easy to check that $\widetilde{W}$ extends the action of $G/Z(G)$ on T.

As T is a locally finite tree, if the edge groups of this splitting are virtually polycyclic then $\widetilde{W} \in \mathfrak{G}(n')$ for some n'. In this case Theorem 3.2 implies that whenever $\widetilde{W}$ is a finite index subgroup of some group $\widetilde{H}$, then $\widetilde{H} \in \mathfrak{G}(n')$ which will provide a G-tree which is fixed by H, the image of $\widetilde{H}$ in $\mathrm{Out}(G)$. We compute the edge stabilizers for the action of $\widetilde{W}$ on T via the following sequences. For an edge $f \subseteq T$ denote by G_f (respectively $\widetilde{W}_f$) the edge stabilizer of f.

Lemma 4.1. *The edge stabilizers of T for the $\widetilde{W}$-action, $\widetilde{W}_f$, fit into short exact sequences:*

$$1 \longrightarrow G_f/Z(G) \longrightarrow \widetilde{W}_f \longrightarrow W_f \longrightarrow 1 \tag{1}$$

where W_f is the image of $\widetilde{W}_f$ in W. In particular, if W_f is virtually polycyclic then $\widetilde{W}_f$ is a virtually polycyclic subgroup of $\widetilde{W}$.

Proof. This only place where exactness needs to be checked is that $G_f/Z(G)$ is the kernel of the map $\widetilde{W}_f \to W_f$. This follows as $\widetilde{W}$ extends the action of G, hence $G/Z(G) \cap \widetilde{W}_f = G_f/Z(G)$. □

We can now prove the main theorem.

Main Theorem. *Suppose $G \in \mathfrak{G}(n)$ and W is a polycyclic subgroup of* $\mathrm{Out}(G)$ *which fixes a point in $\mathcal{D}_G$. If H is a subgroup of* $\mathrm{Out}(G)$ *commensurable with W, then H fixes a point in $\mathcal{D}_G$.*

Proof. Suppose that $H \subseteq \mathrm{Out}(G)$ contains W as a finite index subgroup where W is polycyclic and fixes a point in $\mathcal{D}_G$. We then have the following short exact sequences:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & G/Z(G) & \longrightarrow & \mathrm{Aut}(G) & \longrightarrow & \mathrm{Out}(G) & \longrightarrow & 1 \\ & & \| & & \vee & & \vee & & \\ 1 & \longrightarrow & G/Z(G) & \longrightarrow & \widetilde{H} & \longrightarrow & H & \longrightarrow & 1 \\ & & \| & & \vee & & \vee & & \\ 1 & \longrightarrow & G/Z(G) & \longrightarrow & \widetilde{W} & \longrightarrow & W & \longrightarrow & 1 \end{array} \tag{2}$$

Then as $\widetilde{W} \in \mathfrak{G}(n')$ by Lemma 4.1, and $\widetilde{H}$ contains $\widetilde{W}$ as a finite index subgroup $\widetilde{H} \in \mathfrak{G}(n')$ by Theorem 3.2. Thus $\widetilde{H}$ acts on a locally finite tree T' inducing an action of $G/Z(G)$, hence also G, on T' with virtually polycyclic stabilizers necessarily of Hirsch length n by Lemma 2.3. Let T be the minimal subtree of T' for G. Then $T \in \mathcal{D}_G$ and clearly H fixes this G-tree. □

It seems possible to prove a version of theorem 3.2 to work in the case that W is only finitely generated. We can run the same argument, getting an action of $\widetilde{H}$ on a locally finite tree inducing an action for G. But there is no easy reason why this action should have the correct stabilizers as in Lemma 4.1. However for $n = 0$ or 1, if $G \in \mathfrak{G}(n)$ then for any point in $\mathcal{D}_G$, then the induced graph of groups decomposition has vertex and edge groups with finite outer automorphism groups. Levitt [18] has shown that for these graph of groups decompositions the stabilizer of the decomposition is virtually finitely generated abelian, hence we have the following corollary:

Corollary 4.2. *If $G \in \mathfrak{G}(n)$ for $n = 0$ or 1 and H is a subgroup of* $\mathrm{Out}(G)$ *which contains a finite index subgroup that fixes a point in $\mathcal{D}_G$, then H fixes a point in $\mathcal{D}_G$.*

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