ℓ^2–HOMOLOGY OF THE FREE GROUP

MATT CLAY

Abstract. We analyze the reduced ℓ^2–homology groups of a finitely generated nonabelian free group, $\mathbb{F}$. Specifically, the projection map onto the space of harmonic ℓ^2–1–chains is explicitly described and a weak isomorphism from $(\ell^2(\mathbb{F}))^{\mathrm{rk}(\mathbb{F})-1}$ to the space of harmonic ℓ^2–1–chains is given.

In these notes we collect several calculations regarding the ℓ^2–homology of a finitely generated nonabelian free group. In some places more details regarding standard arguments are provided but no originality is claimed for any of the following material, except for errors.

I became interested in ℓ^2–homology through Mineyev's proof of the Hanna Neumann conjecture [6]; see also [5] for a proof avoiding ℓ^2–homology. In an attempt to understand the tools and techniques of the ℓ^2–theory applied to discrete groups I wanted to see a calculation of the ℓ^2–Betti numbers for a free group using the definition of the von Neumann dimension (this definition appears in Section 3.2). The most direct calculation uses additivity properties of the von Neumann dimension and the Euler characteristic of the free group (this argument appears in Section 3.5). As I was unable to find a calculation using the definition of the von Neumann dimension, I decided to write these notes.

There are excellent surveys of ℓ^2–homology of discrete groups by Eckmann [1] and Lück [3]. Additionally, Lück's book [4] is a very thorough reference on the subject.

I would like to thank Andy Raich for discussions that inspired me to write these notes and to the anonymous referee whose suggestions improve the exposition.

1. Preliminaries

Let $\mathbb{F}$ be a free group with finite rank at least two, denoted $\mathrm{rk}(\mathbb{F})$. We denote the identity element of $\mathbb{F}$ by $\mathbb{1}$. The Hilbert space of square-summable functions $f\colon \mathbb{F} \to \mathbb{C}$ is denoted $\ell^2(\mathbb{F})$. The inner product on $\ell^2(\mathbb{F})$ is given

The author is partially supported by the Simons Foundation.

by:

$$\langle f_1, f_2\rangle = \sum_{g\in\mathbb{F}} f_1(g)\overline{f_2(g)}$$

The dense subset of finitely supported functions is isomorphic (as a vector space) to the group algebra $\mathbb{C}[\mathbb{F}]$. As such, we consider an element $g \in \mathbb{F}$ as the element of $\mathbb{C}[\mathbb{F}] \subset \ell^2(\mathbb{F})$ that is the unit function that takes value 1 on g and 0 elsewhere. The group $\mathbb{F}$ acts on both the left and on the right of $\ell^2(\mathbb{F})$ by:

$$(g \cdot f)(h) = f(g^{-1}h) \text{ and } (f \cdot g)(h) = f(hg^{-1})$$

These extend to left and right actions of $\mathbb{C}[\mathbb{F}]$ by bounded operators. The adjoint of the operator associated to $a \in \mathbb{C}[\mathbb{F}]$ is the corresponding action by the *conjugate* $\overline{a} \in \mathbb{C}[\mathbb{F}]$, where for $f \in \ell^2(\mathbb{F})$ we define $\overline{f}(g) = \overline{f(g^{-1})}$.

We fix a basis $\mathcal{X} \subset \mathbb{F}$ that we will use for the remainder. We denote the Cayley graph of $\mathbb{F}$ with respect to $\mathcal{X}$ by $T = T_{\mathcal{X}}$ and we denote the word length of $g \in \mathbb{F}$ with respect to $\mathcal{X}$ by $|g| = |g|_{\mathcal{X}}$. The set of edges in T is denoted by $E = E(T)$ and the set of vertices by $V = V(T)$. We recall the construction of the Cayley graph for the convenience of the reader. Vertices of T correspond to group elements in $\mathbb{F}$; the vertex of T corresponding to $g \in \mathbb{F}$ is denoted by v_g. There in an edge $e \in E$ with initial vertex $\partial_0(e) = v_g$ and terminal vertex $\partial_1(e) = v_h$ if $h^{-1}g \in \mathcal{X}$, i.e., $g = hx$ for some $x \in \mathcal{X}$. The group $\mathbb{F}$ acts on the left by $gv_h = v_{gh}$. For $x \in \mathcal{X}$, we let ε_x be the edge with endpoints $\partial_0(\varepsilon_x) = v_{\mathbb{1}}$ and $\partial_1(\varepsilon_x) = v_x$ and we set $E_{\mathcal{X}} = \{\varepsilon_x \in E \mid x \in \mathcal{X}\}$. Since $\mathcal{X}$ is a basis, the Cayley graph T is a tree. The local structure of T is shown in Figure 1.

The Hilbert space of square-summable functions $\alpha\colon E \to \mathbb{C}$ is denoted by $\ell^2(E)$. As for $\ell^2(\mathbb{F})$, we will consider an edge $e \in E$ as the unit function in $\ell^2(E)$ that takes value 1 on e and 0 elsewhere. The span of these functions is dense in $\ell^2(E)$ and denoted $\mathbb{C}[E]$. Similarly, the Hilbert space of square-summable functions $\beta\colon V \to \mathbb{C}$ is denoted by $\ell^2(V)$. Again, we will consider vertices as unit functions in $\ell^2(V)$ and denote the span of these functions by $\mathbb{C}[V]$. There is a left action by $\mathbb{F}$ on both $\ell^2(E)$ and $\ell^2(V)$ defined analogously to the action on $\ell^2(\mathbb{F})$: act on the input by the inverse. We remark that the left $\mathbb{F}$–action on $e \in \ell^2(E)$, respectively $v \in \ell^2(V)$, agrees with the $\mathbb{F}$–action on $e \in E$, respectively $v \in V$. Each of these spaces is naturally isomorphic to a product $(\ell^2(\mathbb{F}))^n$ for the appropriate n. Specifically, we have $\ell^2(E) \cong (\ell^2(\mathbb{F}))^{\mathrm{rk}(\mathbb{F})}$ (as there are $|\mathcal{X}| = \mathrm{rk}(\mathbb{F})$ orbits of edges) and $\ell^2(V) \cong \ell^2(\mathbb{F})$ (as there is a single orbit of vertices).

The usual simplicial boundary and co-boundary maps induce bounded $\mathbb{F}$–equivariant operators $\partial\colon \ell^2(E) \to \ell^2(V)$ and $\delta\colon \ell^2(V) \to \ell^2(E)$ defined

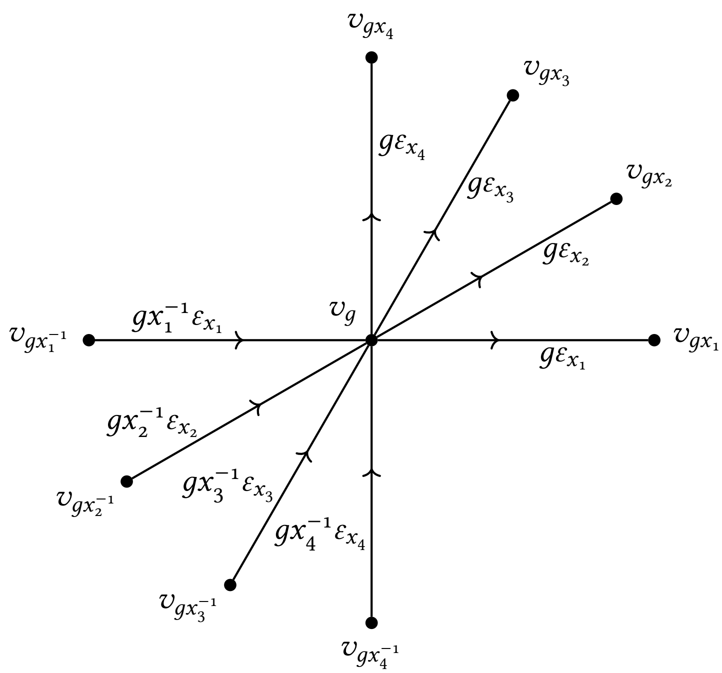

FIGURE 1. The local structure of the Cayley graph T when $\mathcal{X} = \{x_1, x_2, x_3, x_4\}$.

on the unit functions by:

$$\partial e = \partial_1(e) - \partial_0(e),$$
$$\delta v = \sum_{e \in \partial_1^{-1}(v)} e - \sum_{e \in \partial_0^{-1}(v)} e$$

The following lemma shows that these maps are adjoints.

Lemma 1.1. *$\langle \partial\alpha, \beta\rangle = \langle \alpha, \delta\beta\rangle$.*

Proof. For unit functions $e \in \ell^2(E)$ and $v \in \ell^2(V)$ we see that:

$$\langle \partial e, v\rangle = \left\{ \begin{array}{ll} 1 & \text{if } \partial_1(e) = v \\ -1 & \text{if } \partial_0(e) = v \\ 0 & \text{otherwise} \end{array} \right\} = \langle e, \delta v\rangle$$

Since the spans of such functions are dense in the appropriate space, the lemma holds. □

Definition 1.2. The *reduced ℓ^2–homology* groups of $\mathbb{F}$ are:

$$\mathscr{H}_0(\mathbb{F}) = \ker \delta \text{ and } \mathscr{H}_1(\mathbb{F}) = \ker \partial.$$

The purpose of this note is to investigate these groups, specifically, we will compute their von Neumann dimension (see Definition 3.2.2). The von

Neumann dimension of the reduced ℓ^2–homology groups are called the *ℓ^2–Betti numbers* of $\mathbb{F}$ (Definition 3.2.3). First we will show that $\mathcal{H}_0(\mathbb{F}) = 0$ (Theorem 2.1) and so the zeroth ℓ^2–Betti number of $\mathbb{F}$ is 0. More interestingly, we will show that the first ℓ^2–Betti number of $\mathbb{F}$ is $\mathrm{rk}(\mathbb{F}) - 1$ (Theorem 3.2.4), the so-called *reduced rank*. In fact, we will construct a (weak) isomorphism $\left(\ell^2(\mathbb{F})\right)^{\mathrm{rk}(\mathbb{F})-1} \to \mathcal{H}_1(\mathbb{F})$ (Theorem 3.6.3).

Remark 1.3. This fits into a larger framework of the reduced ℓ^2–homology of a chain complex $C_* = \{\partial_n \colon C_n \to C_{n-1}\}$ of Hilbert–Γ–modules. (Here Γ is any discrete group.) In this more general setting the *reduced ℓ^2–homology groups* are defined by:

$$H_n^{(2)}(C_*) = {}^{\ker \partial_n}\Big/_{\mathrm{clos}(\mathrm{im}\, \partial_{n+1})}.$$

This Hilbert space is naturally isomorphic to the Γ–invariant Hilbert subspace of *ℓ^2–harmonic chains*:

$$\mathcal{H}_n(C_*) = \ker \partial_n \cap \ker \delta_n = \ker \Delta_n \subseteq C_n$$

where $\delta_n \colon C_n \to C_{n+1}$ is the adjoint of ∂_n and

$$\Delta_n = \partial_{n+1}\delta_n + \delta_{n-1}\partial_n \colon C_n \to C_n$$

is the *combinatorial Laplacian*. See [1, 4] for more details.

This set-up applies to discrete groups (in particular $\mathbb{F}$) in the following way. Suppose that Γ acts on a contractible CW–complex X by freely permuting the cells of X, such that there are finitely many orbits of cells in each dimension. (This is the situation we have for $\mathbb{F}$ and T.) Let $C_*(X)$ be cellular chain complex of X with coefficients in $\mathbb{C}$. One obtains a chain complex of Hilbert–Γ–modules by tensoring $C_*(X)$ with $\ell^2(\Gamma)$:

$$C_n^{(2)}(X;\ \Gamma) = \ell^2(\Gamma) \otimes_\Gamma C_n(X).$$

The reduced ℓ^2–homology groups of Γ are defined as:

$$H_n^{(2)}(\Gamma) = H_n^{(2)}(C_*^{(2)}(X;\ \Gamma)).$$

The reader can verify that when $\Gamma = \mathbb{F}$, this reduces to the definition given in Definition 1.2. Again, see [1, 4] for more details.

As T is one-dimensional, higher dimensional reduced ℓ^2–homology groups for $\mathbb{F}$ are trivial.

2. The zeroth reduced homology group of $\mathbb{F}$

We begin our investigation into the reduced ℓ^2–homology groups of $\mathbb{F}$ by looking at dimension 0.

Theorem 2.1. $\mathcal{H}_0(\mathbb{F}) = 0$.

Proof. Fix $\beta \in \ell^2(V)$ and suppose that $\delta\beta = 0$. Then for all $g \in \mathbb{F}$ and $x \in \mathcal{X}$ we have $\langle \beta, \partial g \varepsilon_x \rangle = \langle \delta\beta, g\varepsilon_x \rangle = 0$. Thus we find

$$0 = \langle \beta, \partial g\varepsilon_x \rangle = \langle \beta, gxv_{\mathbb{1}} \rangle - \langle \beta, gv_{\mathbb{1}} \rangle = \beta(gxv_{\mathbb{1}}) - \beta(gv_{\mathbb{1}}).$$

Therefore $\beta(gv_{\mathbb{1}}) = \beta(gxv_{\mathbb{1}})$ for all $g \in \mathbb{F}$ and $x \in \mathcal{X}$. Applying this to $gx^{-1} \in \mathbb{F}$, we see $\beta(gx^{-1}v_{\mathbb{1}}) = \beta(gx^{-1}xv_{\mathbb{1}}) = \beta(gv_{\mathbb{1}})$ as well. As $\mathcal{X}$ is a generating set we must have that $\beta(v_{\mathbb{1}}) = \beta(gv_{\mathbb{1}})$ for all $g \in \mathbb{F}$. Since there is a single orbit of vertices, $\beta(v) = 0$ for all $v \in V$ as $\sum_{v \in V} \beta(v)^2 < \infty$. Thus $\beta = 0$.

As δ is injective, $\mathcal{H}_0(\mathbb{F}) = \ker \delta = 0$. □

The proof also shows that $\mathcal{H}_0(\Gamma) = 0$ for any infinite discrete group Γ [1, 4]. The key point is that 0–co-cycles, i.e., functions in $\ker \delta \subseteq \ell^2(\Gamma)$, correspond to constant functions and and the only constant square-summable function on an infinite discrete group is the zero function.

3. The first reduced homology group of $\mathbb{F}$

We now turn our attention to the more interesting situation in dimension 1.

3.1. Examples of cycles in $\mathcal{H}_1(\mathbb{F})$.

To begin, we present a few examples of 1–cycles in $\ell^2(E)$, i.e., functions in $\ker \partial \subseteq \ell^2(E)$.

Definition 3.1.1. Given two edges $e, e' \in E$, we define the *edge distance*, denoted $d(e, e')$ as the number of edges in the edge path starting with e and terminating with e' minus one. If we consider T as a metric space where each edge has length 1, this is the distance between the midpoints of the edges.

Here is our first example of a 1–cycle.

Example 3.1.2. Fix two generators $x_1, x_2 \in \mathcal{X}$ and for simplicity denote the edges ε_{x_1} and ε_{x_2} by ε_1 and ε_2 respectively. For $e \in E$ we set $|e| = d(\varepsilon_1, e)$. Let $W^+ \subset \mathbb{F}$ be the monoid generated by x_1 and x_2 and let $W^- \subset \mathbb{F}$ be the monoid generated by x_1^{-1} and x_2^{-1}. Let

$$S = W^-\{x_1^{-1}\varepsilon_1, x_2^{-1}\varepsilon_2\} \cup \{\varepsilon_1\} \cup x_1 W^+\{\varepsilon_1, \varepsilon_2\} \subset E.$$

The function:

$$\alpha(e) = \begin{cases} 2^{-|e|} & \text{if } e \in S \\ 0 & \text{otherwise} \end{cases}$$

is a 1–cycle.

Indeed, first notice that:

$$\|\alpha\|^2 = \sum_{e\in S} \frac{1}{4^{|e|}} = 1 + 2\sum_{g\in W^+} \frac{1}{4^{|g|+1}} + 2\sum_{g\in W^-} \frac{1}{4^{|g|+1}}$$
$$= 1 + \sum_{g\in W^+} \frac{1}{4^{|g|}} = 1 + \sum_{n=0}^{\infty} 2^n \frac{1}{4^n} = 3$$

so that $\alpha \in \ell^2(E)$. The equality between the first two sums is observed since for $i = 1, 2$, we have $d(\varepsilon_1, x_1 g\varepsilon_i) = |g| + 1$ for $g \in W^+$ and $d(\varepsilon_1, gx_i^{-1}\varepsilon_i) = |g| + 1$ for $g \in W^-$. The equality between the final two sums is observed by summing over the 2^n elements $g \in W^+$ with $|g| = n$.

Considering the function $gv_{\mathbb{1}} \in \ell^2(V)$ we find:

$$\begin{aligned}\langle \partial\alpha, gv_{\mathbb{1}}\rangle &= \langle \alpha, \delta g v_{\mathbb{1}}\rangle \\ &= \sum_{x\in\mathcal{X}} \langle \alpha, gx^{-1}\varepsilon_x\rangle - \langle \alpha, g\varepsilon_x\rangle \\ &= \alpha(gx_1^{-1}\varepsilon_1) + \alpha(gx_2^{-1}\varepsilon_2) - \alpha(g\varepsilon_1) - \alpha(g\varepsilon_2).\end{aligned}$$

There are three cases to treat.

If $g \notin x_1W^+ \cup W^-$, then $\langle \partial\alpha, gv_{\mathbb{1}}\rangle = 0$ as each term is 0.

If $g \in W^-$, we have:

$$\langle \partial\alpha, gv_{\mathbb{1}}\rangle = \frac{1}{2^{|g|+1}} + \frac{1}{2^{|g|+1}} - \frac{1}{2^{|g|}} = 0.$$

Finally if $g = x_1 h$ where $h \in W^+$, we have

$$\langle \partial\alpha, gv_{\mathbb{1}}\rangle = \frac{1}{2^{|g|-1}} - \frac{1}{2^{|g|}} - \frac{1}{2^{|g|}} = 0.$$

Therefore, since the span of the functions $gv_{\mathbb{1}}$ is dense in $\ell^2(V)$, we have that $\partial\alpha = 0$ and so α is indeed a 1–cycle. See Figure 2.

The next example constructs a family of 1–cycles that are important for the calculation of the first ℓ^2–Betti number of $\mathbb{F}$.

Example 3.1.3. Fix a generator $x_i \in \mathcal{X}$ and for simplicity denote ε_{x_i} by ε_i. For an edge $e \in E$, let $|e|_i = d(\varepsilon_i, e)$. We set $\sigma_i(e) = 1$ if e and ε_i are coherently oriented and $\sigma_i(e) = -1$ otherwise. Let $R = 2|\mathcal{X}| - 1$. The function:

$$v_i(e) = \frac{\sigma_i(e)}{R^{|e|_i}} \tag{3.1.1}$$

is a 1–cycle. As before, we first verify that v_i is in $\ell^2(E)$:

$$\|v_i\|^2 = \sum_{e\in E} \frac{1}{R^{2|e|_i}} = 1 + 2\sum_{n=1}^{\infty} \frac{R^n}{R^{2n}} = 1 + \frac{2}{R-1} = \frac{R+1}{R-1} = \frac{|\mathcal{X}|}{|\mathcal{X}| - 1}.$$

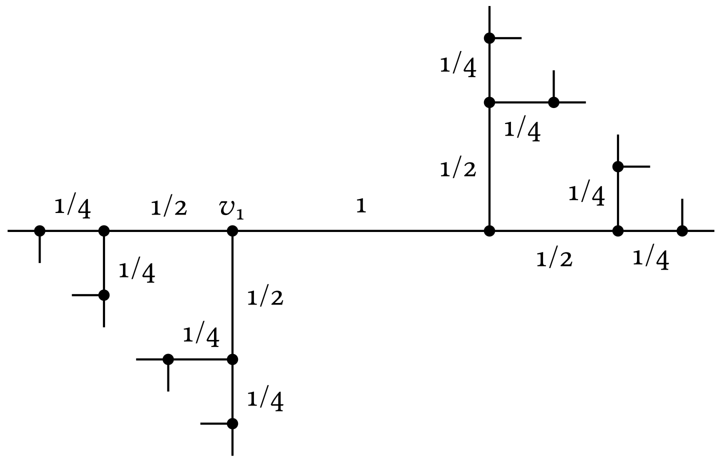

FIGURE 2. The 1–cycle α in Example 3.1.2. At each vertex, the sum of the N and E directions (out-going) equals the sum of the S and W directions (in-coming).

The equality between the second and third sum follows as for $n \geq 1$, there are $2R^n$ edges with $|e|_i = n$. We will now show that $\partial v_i = 0$. Suppose $w \in V$ and let e_0 be the terminal edge in the edge path starting with ε_i and terminating at w. Enumerate the remaining edges adjacent to w by $e_1, \ldots e_R$ where $\sigma_i(e_j) = -\sigma_i(e_0)$ for $j = 1, \ldots, \frac{R-1}{2}$ and $\sigma_i(e_j) = \sigma_i(e_0)$ for $j = \frac{R+1}{2}, \ldots, R$. Notice that for $j = 1, \ldots, R$, we have that $|e_j|_i = |e_0|_i + 1$. Then we find:

$$\langle \partial v_i, w \rangle = \langle v_i, \delta w \rangle = \sum_{j=0}^{\frac{R-1}{2}} \langle v_i, e_j \rangle - \sum_{j=\frac{R+1}{2}}^{R} \langle v_i, e_j \rangle$$

$$= \frac{\sigma(e_0)}{R^{|e_0|}} + \sum_{i=1}^{\frac{R-1}{2}} \frac{-\sigma(e_0)}{R^{|e_0|+1}} - \sum_{i=\frac{R+1}{2}}^{R} \frac{\sigma(e_0)}{R^{|e_0|+1}} = 0.$$

As before, $\partial v_i = 0$ and so v_i is indeed a 1–cycle. See Figure 3.

Definition 3.1.4. The cycle $v_i \in \ell^2(E)$ in (3.1.1) is called the *i–uniform cycle*. We also consider a scaled (not normalized!) version $\Upsilon_i = \frac{1}{\|v_i\|^2} v_i = \frac{|\mathcal{X}|-1}{|\mathcal{X}|} v_i$.

3.2. **von Neumann dimension.** After these examples, we turn our attention to computing the von Neumann dimension of $\mathscr{H}_1(\mathbb{F})$, i.e., the first ℓ^2–Betti number of $\mathbb{F}$.

Before we define the von Neumann dimension, let us pause for a moment and consider how to compute the dimension of a subspace $U \subseteq \mathbb{C}^n$

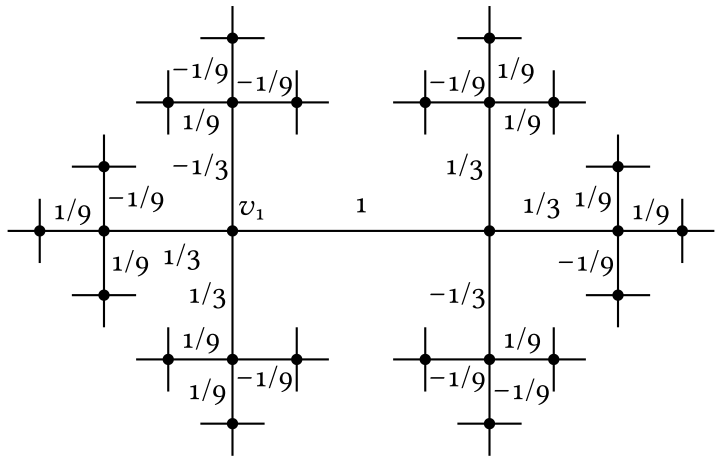

FIGURE 3. The 1−cycle v_1 in Example 3.1.3 when $|\mathcal{X}| = 2$. At each vertex, the sum of the N and E directions (out-going) equals the sum of the S and W directions (in-coming).

without actually finding a basis for U. The tool we will use is orthogonal projection onto U, denoted $\pi_U\colon\ \mathbb{C}^n \to \mathbb{C}^n$. There is a basis $\{b_1, \ldots, b_n\}$ for $\mathbb{C}^n$ such that $\{b_1, \ldots, b_d\}$ is a basis for U (we just care about the existence of such a basis, we do not need to find it). After applying the Gram–Schmidt process, we can assume that $b_{d+1}, \ldots, b_n$ are orthogonal to U. Using this basis, the matrix representing orthogonal projection has block form:

$$\begin{bmatrix} I_d & 0 \\ 0 & 0 \end{bmatrix}$$

where I_d is the $d \times d$ identity matrix. Hence $\mathrm{trace}(\pi_U) = d$, the dimension of U.

The same strategy to define the dimension for (closed) subspaces of $(\ell^2(\mathbb{F}))^n$ will fail since $(\ell^2(\mathbb{F}))^n$ is infinite dimensional as a $\mathbb{C}$−vector space. However, if we make use of the action of $\mathbb{F}$ and restrict ourselves to $\mathbb{F}$−invariant closed subspaces $U \subseteq (\ell^2(\mathbb{F}))^n$, we get an interesting notion of dimension by taking a certain type of trace of the orthogonal projection operator. First we define the von Neumann algebra of $\mathbb{F}$, which is the natural setting for the trace function.

Definition 3.2.1. The *von Neumann algebra* of $\mathbb{F}$, denoted $\mathcal{N}(\mathbb{F})$, is the algebra of bounded (left) $\mathbb{F}$−equivariant operators $\ell^2(\mathbb{F}) \to \ell^2(\mathbb{F})$. The *von Neumann trace* of an operator $m \in \mathcal{N}(\mathbb{F})$ is defined as:

$$\mathrm{trace}_{\mathbb{F}}(m) = \langle m(\mathbb{1}), \mathbb{1}\rangle.$$

The choice of using the identity $\mathbb{1} \in \mathbb{F}$ may seem a bit arbitrary, but the $\mathbb{F}$−equivariance of $m \in \mathcal{N}(\mathbb{F})$ shows that $\langle m(g), g\rangle = \langle m(\mathbb{1}), \mathbb{1}\rangle$ for any

$g \in \mathbb{F}$. Indeed, either notice that $\langle m(g), g\rangle = \langle g \cdot m(\mathbb{1}), g \cdot \mathbb{1}\rangle = \langle m(\mathbb{1}), \mathbb{1}\rangle$ as elements $g \in \mathbb{F}$ act on $\ell^2(\mathbb{F})$ as unitary operators, or explicitly compute:

$$\begin{aligned}\langle m(g), g\rangle &= \big(m(g)\big)(g) = \big(g \cdot m(\mathbb{1})\big)(g) \\ &= \big(m(\mathbb{1})\big)(g^{-1}g) = \big(m(\mathbb{1})\big)(\mathbb{1}) = \langle m(\mathbb{1}), \mathbb{1}\rangle.\end{aligned}$$

An $\mathbb{F}$–invariant bounded operator $M\colon (\ell^2(\mathbb{F}))^n \to (\ell^2(\mathbb{F}))^n$ can be expressed as a matrix $[m_{i,j}]$ of operators $m_{i,j} \in \mathcal{N}(\mathbb{F})$ where:

$$M(f_1, \ldots, f_n) = \left(\sum_{i=1}^{n} m_{1,i}(f_i), \ldots, \sum_{i=1}^{n} m_{n,i}(f_i)\right).$$

We extend the von Neumann trace to such operators in the usual way by considering the diagonal elements:

$$\operatorname{trace}_{\mathbb{F}}(M) = \sum_{i=1}^{n} \operatorname{trace}_{\mathbb{F}}(m_{i,i}).$$

With this motivation and set-up, we can now define the von Neumann dimension.

Definition 3.2.2. Let $\pi_U\colon (\ell^2(\mathbb{F}))^n \to (\ell^2(\mathbb{F}))^n$ denote the orthogonal projection onto a $\mathbb{F}$–invariant Hilbert subspace $U \subseteq (\ell^2(\mathbb{F}))^n$ and express $\pi_U = [m_{i,j}]$ as a matrix of operators $m_{i,j} \in \mathcal{N}(\mathbb{F})$. The *von Neumann dimension* of U is defined as:

$$\dim_{\mathbb{F}}(U) = \operatorname{trace}_{\mathbb{F}}(\pi_U) = \sum_{i=1}^{n} \langle m_{i,i}(\mathbb{1}), \mathbb{1}\rangle. \tag{3.2.1}$$

The von Neumann dimension satisfies several properties akin to the usual dimension:

(1) $\dim_{\mathbb{F}}(M) \geq 0$ and $\dim_{\mathbb{F}}(M) = 0 \iff M = 0$,
(2) if $M \cong N$, then $\dim_{\mathbb{F}}(M) = \dim_{\mathbb{F}}(N)$ and
(3) $\dim_{\mathbb{F}}(M \oplus N) = \dim_{\mathbb{F}}(M) + \dim_{\mathbb{F}}(N)$.

See [4] for details. We remark that the von Neumann dimension is not necessarily an integer but that clearly, $\dim_{\mathbb{F}}\big((\ell^2(\mathbb{F}))^n\big) = n$. Indeed, this follows as the von Neumann trace of the identity operator $I\colon \ell^2(\mathbb{F}) \to \ell^2(\mathbb{F})$ is $\langle I(\mathbb{1}), \mathbb{1}\rangle = \langle \mathbb{1}, \mathbb{1}\rangle = 1$ and hence the the von Neumann trace of the identity operator $I_n\colon (\ell^2(\mathbb{F}))^n \to (\ell^2(\mathbb{F}))^n$ is n.

Using the isomorphism between $(\ell^2(\mathbb{F}))^{\operatorname{rk}(\mathbb{F})}$ and $\ell^2(E)$, if $\pi_U\colon \ell^2(E) \to \ell^2(E)$ is projection onto an $\mathbb{F}$–invariant Hilbert subspace $U \subseteq \ell^2(E)$ then:

$$\dim_{\mathbb{F}}(U) = \operatorname{trace}_{\mathbb{F}}(\pi_U) = \sum_{x \in \mathcal{X}} \langle \pi_U(\varepsilon_x), \varepsilon_x\rangle \tag{3.2.2}$$

Of course, this discussion applies to any discrete group Γ and so $\mathcal{N}(\Gamma)$, trace_Γ and $\dim_\Gamma$ are all well-defined. We can now formally define ℓ^2–Betti numbers.

Definition 3.2.3. Let Γ be a discrete group. The ith *ℓ^2–Betti number* of Γ is $\dim_\Gamma(\mathcal{H}_i(\Gamma))$.

Phrased in this language, Theorem 2.1 states that the zeroth ℓ^2–Betti number of $\mathbb{F}$ (or any infinite discrete group) is 0. We now arrive at what can be thought of as the main result in this note.

Theorem 3.2.4. *The first ℓ^2–Betti number of $\mathbb{F}$ is $\mathrm{rk}(\mathbb{F}) - 1$, in other words, $\dim_\mathbb{F}(\mathcal{H}_1(\mathbb{F})) = \mathrm{rk}(\mathbb{F}) - 1$.*

We will provide two proofs of this theorem in Section 3.5. The first proof is the most direct and often used. It uses the additivity property of von Neumann dimension and the calculation of $\mathcal{H}_0(\mathbb{F})$. The second proof uses the definition of von Neumann dimension in Definition 3.2.2. Specifically, we will give a formula for orthogonal projection onto $\mathcal{H}_1(\mathbb{F}) \subseteq \ell^2(E)$ from which we can compute its trace.

We briefly describe this formula for the orthogonal projection now. For simplicity, in the remainder of this note, we denote $\pi_{\mathcal{H}_1(\mathbb{F})}$ by π and ε_{x_i} by ε_i. The orthogonal projection $\pi\colon\ \ell^2(E) \to \ell^2(E)$ is defined via a type of *convolution* with the component functions of the scaled i–uniform cycles Υ_i of Definition 3.1.4. We therefore need a sufficient condition for when convolving with a fixed element of $\ell^2(\mathbb{F})$ defines an element of the von Neumann algebra $\mathcal{N}(\mathbb{F})$. This is in the content of the next Section 3.3.

Following this, we will show in Section 3.4 that $\pi\colon\ \ell^2(E) \to \ell^2(E)$ satisfies

$$\langle \pi\varepsilon_i, g\varepsilon_j\rangle = \Upsilon_i(g^{-1}\varepsilon_i).$$

where Υ_i is the scaled i–uniform cycle from Definition 3.1.4. As $\Upsilon_i(\varepsilon_i) = \frac{\mathrm{rk}(\mathbb{F})-1}{\mathrm{rk}(\mathbb{F})}$, this will show in Section 3.5 that $\dim_\mathbb{F}(\mathcal{H}_1(\mathbb{F})) = \mathrm{rk}(\mathbb{F}) - 1$.

3.3. Convolution in $\ell^2(\mathbb{F})$.

In this section we give a sufficient condition for the convolution of two square-summable functions to be square-summable. A similar condition was proved by Haagerup [2] and would suffice for our purposes. For completeness, we provide a proof.

Definition 3.3.1. A function $\tau\colon\ \mathbb{F} \to \mathbb{C}$ is *exponentially decaying* if there exists a constant C such that $|\tau(g)| \leq \frac{C}{(2\,\mathrm{rk}(\mathbb{F})-1)^{|g|}}$ for all $g \in \mathbb{F}$.

Notice that a exponentially decreasing function is square-summable, but not necessarily summable.

Proposition 3.3.2. *Let $\tau\colon\ \mathbb{F} \to \mathbb{C}$ be an exponentially decaying function. Then for any $f \in \ell^2(\mathbb{F})$ the convolution $f * \tau\colon\ \mathbb{F} \to \mathbb{C}$ defined by:*

$$(f * \tau)(g) = \sum_{h\in\mathbb{F}} f(gh^{-1})\tau(h) \tag{3.3.1}$$

is a square-summable function, i.e., $f * \tau \in \ell^2(\mathbb{F})$. Moreover, there is a constant $B = B(C, \mathrm{rk}(\mathbb{F}))$ such that $\|f * \tau\| \leq B\|f\|$.

Proof. Let $R = 2\,\mathrm{rk}(\mathbb{F}) - 1$ and let C be the constant such that $|\tau(g)| \leq \frac{C}{R^{|g|}}$ for all $g \in \mathbb{F}$. We will first produce a constant C_0 such that $|(f * \tau)(g)| \leq \frac{C_0(|g|+1)}{R^{|g|}}$ for all $g \in \mathbb{F}$.

Given two words $g, h \in \mathbb{F}$ we denote the common prefix by $g \wedge h$. For $g \in \mathbb{F}$ we set $P_g^n = \{h \in \mathbb{F} \mid |g \wedge h| = n\}$. These sets form a partition: $\mathbb{F} = \cup_{n=0}^{|g|} P_g^n$. Therefore we can write:

$$\begin{aligned}
|(f * \tau)(g)| &\leq \sum_{h\in\mathbb{F}} |f(gh^{-1})\tau(h)| = \sum_{h\in\mathbb{F}} |f(h)\tau(h^{-1}g)| \\
&= \sum_{n=0}^{|g|} \sum_{h\in P_g^n} |f(h)\tau(h^{-1}g)| \\
&\leq \sum_{n=0}^{|g|} C \sum_{h\in P_g^n} \frac{|f(h)|}{R^{|h|+|g|-n}} = \sum_{n=0}^{|g|} \frac{C}{R^{|g|}} \sum_{h\in P_g^n} \frac{|f(h)|}{R^{|h|-n}}.
\end{aligned} \tag{3.3.2}$$

For each $n \in \{0, \ldots, |g|\}$ we let $c_n\colon\ \mathbb{F} \to \{0, 1\}$ be the characteristic function of the set P_g^n. Define $f_n(h) = c_n(h)f(h)$ and $r_n(h) = \frac{c_n(h)}{R^{|h|-n}}$. Clearly $\|f_n\| \leq \|f\|$. As for r_n we find:

$$\|r_n\|^2 = \sum_{h\in P_g^n} \frac{1}{R^{2(|h|-n)}} = \sum_{k=0}^{\infty} \sum_{\substack{|h|=n+k \\ h\in P_g^n}} \frac{1}{R^{2k}} \leq \sum_{k=0}^{\infty} R^k \frac{1}{R^{2k}} = \frac{R}{R-1}.$$

Therefore, by the Cauchy–Schwartz inequality we have:

$$\sum_{h\in P_g^n} \frac{|f(h)|}{R^{|h|-n}} = \sum_{h\in\mathbb{F}} |f_n(h)r_n(h)| \leq \|f_n\| \cdot \|r_n\| \leq \|f\| \left(\frac{R}{R-1}\right)^{1/2}$$

Combining this with the the inequality in (3.3.2) we have:

$$
\begin{aligned}
|(f * \tau)(g)| &\leq \sum_{n=0}^{|g|} \frac{C}{R^{|g|}} \sum_{h \in P_g^n} \frac{|f(h)|}{R^{|h|-n}} \\
&\leq \frac{C(|g|+1)}{R^{|g|}} \|f\| \left(\frac{R}{R-1} \right)^{1/2} \\
&\leq \frac{C_0(|g|+1)}{R^{|g|}}
\end{aligned}
$$

as sought where $C_0 = C\|f\| \left(\frac{R}{R-1}\right)^{1/2}$. We then find:

$$
\begin{aligned}
\sum_{g \in \mathbb{F}} |(f * \tau)(g)|^2 &\leq C_0^2 \sum_{g \in \mathbb{F}} \frac{(|g|+1)^2}{R^{2|g|}} \\
&= C_0^2 \sum_{n=0}^{\infty} \sum_{|g|=n} \frac{(|g|+1)^2}{R^{2|g|}} \\
&\leq C_0^2 \sum_{n=0}^{\infty} (R+1) R^{n-1} \frac{(n+1)^2}{R^{2n}} \\
&\leq C_0^2 \left(\frac{R+1}{R} \right) \sum_{n=0}^{\infty} \frac{(n+1)^2}{R^n} = C_0^2 \left(\frac{R+1}{R} \right) \frac{4R^2 - 3R + 1}{(R-1)^3}
\end{aligned}
$$

as desired. The existence of a constant $B = B(C, \mathrm{rk}(F))$ such that $\|f * \tau\| \leq B\|f\|$ is now clear as C_0 is a constant times $\|f\|$. $\square$

We record the following immediate properties of convolution.

Proposition 3.3.3. *Let $\tau \colon \mathbb{F} \to \mathbb{C}$ be an exponentially decaying function. Then:*

(1) $\forall f_1, f_2 \in \ell^2(\mathbb{F})$: $(f_1 + f_2) * \tau = f_1 * \tau + f_2 * \tau$,
(2) $\forall f \in \ell^2(\mathbb{F})$, $\lambda \in \mathbb{C}$: $(\lambda f) * \tau = \lambda(f * \tau) = f * (\lambda \tau)$,
(3) $\forall f \in \ell^2(\mathbb{F})$, $g \in \mathbb{F}$: $g \cdot (f * \tau) = (g \cdot f) * \tau$, *and*
(4) $\forall g, h \in \mathbb{F}$: $(h * \tau)(g) = \tau(h^{-1}g)$.

3.4. **The projection operator $\pi_{\mathcal{H}_1(\mathbb{F})}$.** As stated in Section 3.2, we denote the orthogonal projection $\pi_{\mathcal{H}_1(\mathbb{F})} \colon \ell^2(E) \to \ell^2(E)$ by π and the edges $\varepsilon_{x_i} \in E_{\mathfrak{X}}$ by ε_i. Further, for the remainder, set $R = 2\,\mathrm{rk}(\mathbb{F}) - 1$.

The main result of this section is an explicit description of π. This is the key ingredient of the second proof of Theorem 3.2.4 presented in Section 3.5.

Theorem 3.4.1. *If $\alpha \in \ell^2(E)$, then $\pi\alpha(g\varepsilon_i) = \langle g^{-1} \cdot \alpha, \Upsilon_i \rangle$.*

We first show that the formula in Theorem 3.4.1 defines an $\mathbb{F}$−equivariant bounded operator on $\ell^2(E)$. Verifying that the formula does indeed describe orthogonal projection occurs in a series of lemmas following the proposition.

Proposition 3.4.2. *Suppose $\alpha \in \ell^2(E)$ and $i = 1, \ldots, \mathrm{rk}(\mathbb{F})$.*

(1) *The function $f\colon \mathbb{F} \to \mathbb{C}$ defined by $f(g) = \langle g^{-1} \cdot \alpha, \Upsilon_i\rangle$ is in $\ell^2(\mathbb{F})$.*
(2) *The function $\Upsilon\alpha\colon E \to \mathbb{C}$ defined by $\Upsilon\alpha(g\varepsilon_i) = \langle g^{-1} \cdot \alpha, \Upsilon_i\rangle$ is in $\ell^2(E)$.*
(3) *The assignment $\alpha \mapsto \Upsilon\alpha$ defines a bounded $\mathbb{F}$−equivariant operator on $\ell^2(E)$.*

Proof. For $j = 1, \ldots, \mathrm{rk}(\mathbb{F})$, define component functions $\alpha_j, \Upsilon_{i,j}\colon \mathbb{F} \to \mathbb{C}$ by $\alpha_j(g) = \alpha(g\varepsilon_j)$ and $\Upsilon_{i,j}(g) = \Upsilon_i(g\varepsilon_j)$. Then $\alpha_j, \Upsilon_{i,j} \in \ell^2(\mathbb{F})$ and $|\Upsilon_{i,j}(g)| \le \frac{1}{R^{|g|}}$ and hence is exponentially decreasing. Then:

$$\begin{aligned} f(g) = \langle g^{-1} \cdot \alpha, \Upsilon_i\rangle &= \sum_{e\in E} \alpha(ge)\Upsilon_i(e) \\ &= \sum_{j=1}^{\mathrm{rk}(\mathbb{F})} \sum_{h\in\mathbb{F}} \alpha(gh\varepsilon_j)\Upsilon_i(h\varepsilon_j) = \sum_{j=1}^{\mathrm{rk}(\mathbb{F})} \sum_{h\in\mathbb{F}} \alpha_j(gh)\Upsilon_{i,j}(h) \\ &= \sum_{j=1}^{\mathrm{rk}(\mathbb{F})} (\alpha_j * \overline{\Upsilon_{i,j}})(g). \end{aligned}$$

By Proposition 3.3.2, we have that each function $\alpha_j * \overline{\Upsilon_{i,j}}$ is square-summable, hence so is f. This proves (1). Since we can view $\Upsilon\alpha$ as a tuple of functions as in (1), this proves (2) as well.

That Υ defines a bounded operator follows from Proportions 3.3.2 and 3.3.3 and the above description. Equivariance follows from the observation:

$$\Upsilon(h \cdot \alpha)(g\varepsilon_i) = \langle g^{-1}h \cdot \alpha, \Upsilon_i\rangle = \Upsilon\alpha(h^{-1}g\varepsilon_i) = h \cdot (\Upsilon\alpha)(g\varepsilon_i).$$

This proves (3). ◻

The next three lemmas will show that $\Upsilon = \pi$, where $\pi\colon \ell^2(E) \to \ell^2(E)$ is orthogonal projection onto $\mathcal{H}_1(\mathbb{F})$.

Lemma 3.4.3. *If $\beta \in \ell^2(V)$ then $\Upsilon(\delta\beta) = 0$.*

Proof. $\Upsilon(\delta\beta)(g\varepsilon_i) = \langle g^{-1} \cdot \delta\beta, \Upsilon_i\rangle = \langle \delta(g^{-1} \cdot \beta), \Upsilon_i\rangle = \langle g^{-1} \cdot \beta, \partial\Upsilon_i\rangle = 0$. ◻

Hence $\overline{\mathrm{im}\,\delta} \subseteq \ker \Upsilon$. The following averaging estimation is the key ingredient for showing that Υ is the identity on cycles.

Lemma 3.4.4. *If $\alpha \in \ell^2(E)$ and $\partial\alpha = 0$, then for all $g \in \mathbb{F}$, $i = 1, \dots, \mathrm{rk}(\mathbb{F})$ and $N \geq 0$:*

$$\sum_{n=0}^{N} \sum_{|e|_i = n} \alpha(ge)v_i(e) = \alpha(g\varepsilon_i)\left(1 + 2\sum_{n=1}^{N} \frac{1}{R^n}\right). \tag{3.4.1}$$

Proof. To begin we claim that for $n \geq 2$:

$$\sum_{|e|_i = n} \alpha(ge)v_i(e) = \frac{1}{R} \sum_{|e|_i = n-1} \alpha(ge)v_i(e) \tag{3.4.2}$$

To see this, fix an edge e with $|e|_i = n - 1$ and enumerate the R edges $e_1, \dots, e_R$ adjacent to e with $|e_j|_i = n$ for $j = 1, \dots, R$ where $\sigma(e_j) = -\sigma(e)$ for $j = 1, \dots, \frac{R-1}{2}$ and $\sigma(e_j) = \sigma(e)$ for $j = \frac{R+1}{2}, \dots, R$. Since $\partial\alpha = 0$ we have:

$$\alpha(ge) = \sum_{j=\frac{R+1}{2}}^{R} \alpha(ge_j) - \sum_{j=1}^{\frac{R-1}{2}} \alpha(ge_j) = \sum_{j=1}^{R} \sigma(e_j)\alpha(ge_j).$$

Multiplying this equation by $\frac{\sigma(e)}{R^{|e|_i+1}} = \frac{1}{R}v_i(e) = \frac{1}{\sigma(e_j)}v_i(e_j)$ we get:

$$\frac{1}{R}\alpha(ge)v_i(e) = \sum_{j=1}^{R} \alpha(ge_j)v_i(e_j) \tag{3.4.3}$$

Equation (3.4.2) follows from (3.4.3) by summing over all edges with $|e|_i = n - 1$.

Then using induction on (3.4.2) we find:

$$\sum_{|e|_i = n} \alpha(ge)v_i(e) = \frac{1}{R^{n-1}} \sum_{|e|_i = 1} \alpha(ge)v_i(e).$$

The above argument for (3.4.2) also shows that the summation on the right is exactly $\frac{2}{R}\alpha(g\varepsilon_i)v_i(\varepsilon_i) = \frac{2}{R}\alpha(g\varepsilon_i)$, the difference being that there are $2R$ edges adjacent to ε_i whose edge distance to ε_i is 1 and that (3.4.3) holds for each of the two subsets of size R. Hence for $n \geq 1$ we have:

$$\sum_{|e|_i = n} \alpha(ge)v_i(e) = \frac{2}{R^n}\alpha(g\varepsilon_i)$$

and so (3.4.1) holds. ∎

Lemma 3.4.5. *If $\alpha \in \ell^2(E)$ and $\partial\alpha = 0$, then $\Upsilon\alpha = \alpha$.*

Proof. Using Lemma 3.4.4 we find for any $g \in \mathbb{F}$ and $i = 1, \ldots, \mathrm{rk}(\mathbb{F})$:

$$\begin{aligned}
\Upsilon\alpha(g\varepsilon_i) = \langle g^{-1} \cdot \alpha, \Upsilon_i \rangle &= \sum_{e \in E} \alpha(ge)\Upsilon_i(e) \\
&= \left(\frac{R-1}{R+1}\right) \sum_{n=0}^{\infty} \sum_{|e|_i = n} \alpha(ge) v_i(e) \\
&= \alpha(g\varepsilon_i) \left(\frac{R-1}{R+1}\right) \left(1 + 2\sum_{n=1}^{\infty} \frac{1}{R^n}\right) \\
&= \alpha(g\varepsilon_i) \left(\frac{R-1}{R+1}\right) \left(1 + \frac{2}{R-1}\right) = \alpha(g\varepsilon_i). \qquad \square
\end{aligned}$$

From these lemmas, it follows that Υ is orthogonal projection onto $\ker \partial$.

Proof of Theorem 3.4.1. Lemma 3.4.3 shows that $\overline{\operatorname{im}\delta} \subseteq \ker \Upsilon$ and Lemma 3.4.5 shows that Υ is the identity on $\ker \partial$. As $\ell^2(E) = \ker \partial \perp \overline{\operatorname{im}\delta}$ we have that Υ is orthogonal projection onto $\ker \partial = \mathscr{H}_1(\mathbb{F})$. □

3.5. **Two proofs of Theorem 3.2.4.** We now give two calculations showing that the first ℓ^2–Betti number of $\mathbb{F}$ is $\mathrm{rk}(\mathbb{F}) - 1$; this is the content of Theorem 3.2.4.

First proof of Theorem 3.2.4. The adjoint relation between ∂ and δ gives a orthogonal direct sum decomposition:

$$\ell^2(E) = \ker \partial \perp \overline{\operatorname{im}\delta}.$$

By Theorem 2.1 we have that δ is injective and so $\dim_{\mathbb{F}}(\overline{\operatorname{im}\delta}) = \dim_{\mathbb{F}}(\ell^2(\mathbb{F})) = 1$. Thus:

$$\dim_{\mathbb{F}}(\mathscr{H}_1(\mathbb{F})) = \dim_{\mathbb{F}}(\ker \partial) = \dim_{\mathbb{F}}(\ell^2(E)) - \dim_{\mathbb{F}}(\overline{\operatorname{im}\delta}) = \mathrm{rk}(\mathbb{F}) - 1. \qquad \square$$

Second proof of Theorem 3.2.4. We have $\pi\varepsilon_j(g\varepsilon_i) = \langle g^{-1}\varepsilon_j, \Upsilon_i \rangle = \Upsilon_i(g^{-1}\varepsilon_j)$ by Theorem 3.4.1. Using (3.2.2) we have:

$$\begin{aligned}
\dim_{\mathbb{F}}(\mathscr{H}_1(\mathbb{F})) = \operatorname{trace}_{\mathbb{F}}(\pi) &= \sum_{i=1}^{\mathrm{rk}(\mathbb{F})} \langle \pi\varepsilon_i, \varepsilon_i \rangle = \sum_{i=1}^{\mathrm{rk}(\mathbb{F})} \Upsilon_i(\varepsilon_i) \\
&= \sum_{i=1}^{\mathrm{rk}(\mathbb{F})} \frac{\mathrm{rk}(\mathbb{F}) - 1}{\mathrm{rk}(\mathbb{F})} = \mathrm{rk}(\mathbb{F}) - 1. \qquad \square
\end{aligned}$$

3.6. **Weak isomorphisms between $(\ell^2(\mathbb{F}))^{\mathrm{rk}(\mathbb{F})-1}$ and $\mathscr{H}_1(\mathbb{F})$.** We have now seen that $\dim_{\mathbb{F}}(\mathscr{H}_1(\mathbb{F})) = \mathrm{rk}(\mathbb{F}) - 1$. We conclude our analysis of $\mathscr{H}_1(\mathbb{F})$ by explicitly describing a weak isomorphism $(\ell^2(\mathbb{F}))^{\mathrm{rk}(\mathbb{F})-1} \to \mathscr{H}_1(\mathbb{F})$. Recall, a bounded $\mathbb{F}$–equivariant operator $U \to V$ of Hilbert–$\mathbb{F}$–modules is a *weak isomorphism* if it is injective with dense image. Using polar decomposition, it can be shown that two weakly isomorphic Hilbert–$\mathbb{F}$–modules are isometrically isomorphic [1, Lemma 2.5.3].

The weak isomorphism we are interested in is the adjoint of the projection onto the first $\mathrm{rk}(\mathbb{F})-1$ component functions in $(\ell^2(\mathbb{F}))^{\mathrm{rk}(\mathbb{F})}$.

Theorem 3.6.1. *The operator* $P\colon (\ell^2(\mathbb{F}))^{\mathrm{rk}(\mathbb{F})} \to (\ell^2(\mathbb{F}))^{\mathrm{rk}(\mathbb{F})-1}$ *defined by:*

$$P(f_1,\ldots,f_{\mathrm{rk}(\mathbb{F})}) = (f_1,\ldots,f_{\mathrm{rk}(\mathbb{F})-1}) \tag{3.6.1}$$

restricts to a weak isomorphism $P\colon \mathcal{H}_1(\mathbb{F}) \to (\ell^2(\mathbb{F}))^{\mathrm{rk}(\mathbb{F})-1}$.

Proof. As $\dim_{\mathbb{F}}(\mathcal{H}_1(\mathbb{F})) = \mathrm{rk}(\mathbb{F}) - 1 = \dim_{\mathbb{F}}((\ell^2(\mathbb{F}))^{\mathrm{rk}(\mathbb{F})-1})$, it suffices to show that P is injective on $\mathcal{H}_1(\mathbb{F})$ [4, Lemma 1.13].

To this end, we suppose $\alpha = (0,\ldots,0,f) \in \mathcal{H}_1(\mathbb{F})$, in other words, $\alpha \in \ker P \cap \ker \partial$. Then for each $g \in \mathbb{F}$ we have

$$0 = \langle \partial\alpha, gv_{\mathbb{1}}\rangle = \langle \alpha, \delta g v_{\mathbb{1}}\rangle = f(gx_{\mathrm{rk}(\mathbb{F})}^{-1}) - f(g).$$

From this it follows that $f(g) = f(gx^n_{\mathrm{rk}(\mathbb{F})})$ for all $n \in \mathbb{Z}$. Since

$$\sum_{n\in\mathbb{Z}} f(gx^n_{\mathrm{rk}(\mathbb{F})})^2 \leq \|f\|^2 < \infty,$$

we must have $f(g) = 0$. As $g \in \mathbb{F}$ was arbitrary, we have $f = 0$ and so $\alpha = 0$. Thus $\ker P \cap \ker \partial = \{0\}$ and so P is injective on $\mathcal{H}_1(\mathbb{F})$. □

Remark 3.6.2. The operator $P\colon \mathcal{H}_1(\mathbb{F}) \to (\ell^2(\mathbb{F}))^{\mathrm{rk}(\mathbb{F})-1}$ is not surjective. For instance $(1,0,\ldots,0) \notin \operatorname{im} P$. Indeed, if $(1,0,\ldots,0,f) \in \mathcal{H}_1(\mathbb{F}) = \ker\partial$ then arguing as in the proof of Theorem 3.6.1 we see that for all $n > 0$ that

$$f(x^{-n}_{\mathrm{rk}(\mathbb{F})}) = f(x^{-1}_{\mathrm{rk}(\mathbb{F})}) = f(1) + 1 = f(x^n_{\mathrm{rk}(\mathbb{F})}) + 1.$$

Such a function could not be square-summable.

We now seek to describe the adjoint $P^*\colon (\ell^2(\mathbb{F}))^{\mathrm{rk}(\mathbb{F})-1} \to \mathcal{H}_1(\mathbb{F})$. As in the proof of Proposition 3.4.2, for $i,j = 1,\ldots,\mathrm{rk}(\mathbb{F})$ we define $\Upsilon_{i,j} \in \ell^2(\mathbb{F})$ by $\Upsilon_{i,j}(g) = \Upsilon_i(g\varepsilon_j)$ where $\Upsilon_i \in \ell^2(E)$ is the scaled i–uniform cycle from Definition 3.1.4. Then for $\beta = (\beta_1,\ldots,\beta_{\mathrm{rk}(\mathbb{F})-1}) \in (\ell^2(\mathbb{F}))^{\mathrm{rk}(\mathbb{F})-1}$ we define

$$Q_j\beta = \sum_{i=1}^{\mathrm{rk}(\mathbb{F})-1} \beta_i * \Upsilon_{i,j}.$$

We recall for convenience in the proof of Theorem 3.6.3 that

$$\beta_i * \Upsilon_{i,j}(h) = \sum_{g\in\mathbb{F}} \beta_i(hg)\Upsilon_{i,j}(g^{-1}) = \sum_{g\in\mathbb{F}} \beta_i(hg)\overline{\Upsilon_{i,j}}(g) = \langle h^{-1}\cdot\beta_i, \overline{\Upsilon_{i,j}}\rangle.$$

Then as in Proposition 3.4.2, $Q_j\colon (\ell^2(\mathbb{F}))^{\mathrm{rk}(\mathbb{F})-1} \to \ell^2(\mathbb{F})$ is a bounded $\mathbb{F}$–equivariant operator.

Theorem 3.6.3. *The adjoint $P^*\colon (\ell^2(\mathbb{F}))^{\mathrm{rk}(\mathbb{F})-1} \to \mathscr{H}_1(\mathbb{F})$ is given by:*

$$P^*\beta = (Q_1\beta, \ldots, Q_{\mathrm{rk}(\mathbb{F})}\beta) \tag{3.6.2}$$

and is a weak isomorphism.

Proof. The adjoint of a weak isomorphism is always weak isomorphism, so the content of the theorem is that the formula in (3.6.2) defines P^*. This is a direct calculation essentially reproducing the fact the adjoint of convolving against a function f is convolving against the conjugate $\overline{f}$.

Fix $\alpha = (\alpha_1, \ldots, \alpha_{\mathrm{rk}(\mathbb{F})}) \in \mathscr{H}_1(\mathbb{F})$ and $\beta = (\beta_1, \ldots, \beta_{\mathrm{rk}(\mathbb{F})-1}) \in (\ell^2(\mathbb{F}))^{\mathrm{rk}(\mathbb{F})-1}$. By Theorem 3.4.1 we have that $\pi\alpha = \alpha$ and so:

$$\alpha_i(g) = \alpha(g\varepsilon_i) = \pi\alpha(g\varepsilon_i) = \langle g^{-1}\cdot\alpha, \Upsilon_i\rangle = \sum_{j=1}^{\mathrm{rk}(\mathbb{F})} \langle \alpha_j, g\cdot\Upsilon_{i,j}\rangle.$$

Using this we compute:

$$\begin{aligned}
\langle P\alpha, \beta\rangle &= \sum_{i=1}^{\mathrm{rk}(\mathbb{F})-1} \langle \alpha_i, \beta_i\rangle = \sum_{i=1}^{\mathrm{rk}(\mathbb{F})-1} \sum_{g\in\mathbb{F}} \alpha_i(g)\overline{\beta_i(g)} \\
&= \sum_{i=1}^{\mathrm{rk}(\mathbb{F})-1} \sum_{g\in\mathbb{F}} \left(\sum_{j=1}^{\mathrm{rk}(\mathbb{F})} \langle \alpha_j, g\cdot\Upsilon_{i,j}\rangle \right) \overline{\beta_i(g)} \\
&= \sum_{i=1}^{\mathrm{rk}(\mathbb{F})-1} \sum_{j=1}^{\mathrm{rk}(\mathbb{F})} \sum_{g\in\mathbb{F}} \left(\sum_{h\in\mathbb{F}} \alpha_j(h)\overline{\Upsilon_{i,j}(g^{-1}h)} \right) \overline{\beta_i(g)} \\
&= \sum_{j=1}^{\mathrm{rk}(\mathbb{F})} \sum_{i=1}^{\mathrm{rk}(\mathbb{F})-1} \sum_{h\in\mathbb{F}} \alpha_j(h) \left(\sum_{g\in\mathbb{F}} \overline{\Upsilon_{i,j}}(h^{-1}g)\overline{\beta_i(g)} \right) \\
&= \sum_{j=1}^{\mathrm{rk}(\mathbb{F})} \sum_{i=1}^{\mathrm{rk}(\mathbb{F})-1} \sum_{h\in\mathbb{F}} \alpha_j(h)\overline{\langle \beta_i, h\cdot\overline{\Upsilon_{i,j}}\rangle} \\
&= \sum_{j=1}^{\mathrm{rk}(\mathbb{F})} \sum_{i=1}^{\mathrm{rk}(\mathbb{F})-1} \langle \alpha_j, \beta_i * \Upsilon_{i,j}\rangle = \sum_{j=1}^{\mathrm{rk}(\mathbb{F})} \langle \alpha_j, Q_j\beta\rangle = \langle \alpha, P^*\beta\rangle.
\end{aligned}$$

This proves the theorem. ∎

References

[1] B. Eckmann, *Introduction to l_2-methods in topology: reduced l_2-homology, harmonic chains, l_2-Betti numbers*, Israel J. Math., 117 (2000), pp. 183–219. Notes prepared by Guido Mislin.

[2] U. Haagerup, *An example of a nonnuclear C^*-algebra, which has the metric approximation property*, Invent. Math., 50 (1978/79), pp. 279–293.

[3] W. Lück, *L^2-invariants of regular coverings of compact manifolds and CW-complexes*, in Handbook of geometric topology, North-Holland, Amsterdam, 2002, pp. 735–817.
[4] ———, *L^2-invariants: theory and applications to geometry and K-theory*, vol. 44 of Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics [Results in Mathematics and Related Areas. 3rd Series. A Series of Modern Surveys in Mathematics], Springer-Verlag, Berlin, 2002.
[5] I. Mineyev, *Groups, graphs, and the Hanna Neumann conjecture*, J. Topol. Anal., 4 (2012), pp. 1–12.
[6] ———, *Submultiplicativity and the Hanna Neumann conjecture*, Ann. of Math. (2), 175 (2012), pp. 393–414.

Dept. of Mathematics, University of Arkansas, Fayetteville, AR 72701
E-mail address: mattclay@uark.edu