Geometric Group Theory

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Introduction

Groups and spaces go hand in hand. For a given space, there are many groups associated to it. We can consider the group of symmetries, that is, the group of structure preserving bijections. Additionally, there is the fundamental group and also the homology and cohomology groups to name a few more. As pointed out by Hermann Weyl, these groups can give "a deep insight" into a given space. An example of this phenomenon is in the study of knots. Algebraic invariants in the form of groups show that the trefoil knot cannot be unknotted for instance. See Figure 1.

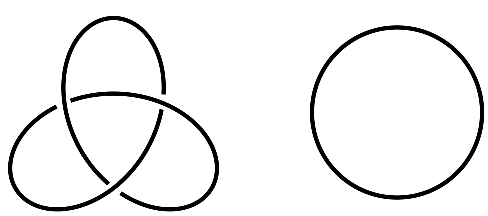

Figure 1: Groups show that these knots are distinct.

Geometric group theory takes a different perspective on this relationship between groups and spaces. Rather than using the algebraic structure and properties of groups to study spaces, the main philosophy of geometric group theory is the following.

> *Study groups using the topology and geometry of the spaces they act on.*

That is, groups are the central objects of study and the techniques and tools used to investigate them are dynamical, geometrical, and topological in nature.

In name, geometric group theory is quite new in relation to other mathematical fields[1]. The foundational essays by Gromov [Gro87, Gro93] introducing the notion of hyperbolic groups and initiating the study of finitely generated groups as metric spaces sparked an enormous amount of research and established lines of investigation that are still very active today. Prior to the emergence of geometric group theory, there were geometrical ideas present in group theory in the works of Dehn, Whitehead, van Kampen and others. Additionally, Thurston's work on 3–manifolds showed how the geometry of a manifold influences algebraic and algorithmic properties of its fundamental group. It is Gromov's essays though that mark the beginning of where these ideas are the forefront.

This article is intended to give an idea about how the topology and geometry of a space influences the algebraic structure of groups that act on it and how this can be used to investigate groups. As you will see, I take the approach I learned from my advisor Mladen Bestvina of favoring illustrative examples over general theory. As is true of any survey of a mathematical field, many aspects and areas of geometric group theory are not mentioned at all. The final section includes a short list of books on geometric group theory for further reading.

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[1]The earliest use of the "geometric group theory" I could find was in reference to a symposium at Sussex University in the summer of 1991.

Groups and spaces

As mentioned above, geometric group theory uses group actions on spaces to understand the group's structure. What type of information could one hope to glean from an action? Are there always interesting actions to study? We will take a look at both of these questions now.

An example: $\mathbf{SL(2,\mathbb{Z})}$

To give an illustration how the topology of a space that a group acts on influences the group's structure, let's take a look at an example of a group action that appears in many areas of mathematics. We will consider the group of 2×2 matrices with integer entries and determinant equal to 1. This group is called *the special linear group*:

$$\mathrm{SL}(2,\mathbb{Z}) = \left\{ \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \mid a,b,c,d \in \mathbb{Z} \text{ and } ad-bc=1 \right\}.$$

Is $\mathrm{SL}(2,\mathbb{Z})$ finitely generated? That is, are there finitely many matrices $A_1,\dots,A_n \in \mathrm{SL}(2,\mathbb{Z})$ such that any matrix $M \in \mathrm{SL}(2,\mathbb{Z})$ can be expressed as a product $M = A_{j_1}^{\pm 1} \cdots A_{j_k}^{\pm 1}$? (Note, each A_j may appear multiple times.) The answer is "yes" and there is an algebraic approach to this problem, but let's take a geometric perspective and consider an action of $\mathrm{SL}(2,\mathbb{Z})$ on a metric space.

The space we will consider is the *Farey complex* which is constructed as follows. First, we start with a graph whose vertex set is the set of rational numbers $\frac{p}{q}$—always expressed in lowest terms—along with an additional point we denote $\frac{1}{0}$. Edges join two vertices $\frac{p}{q}$ and $\frac{r}{s}$ if $ps - qr = \pm 1$. Figure 2 shows a portion of this graph, known as the *Farey graph.*

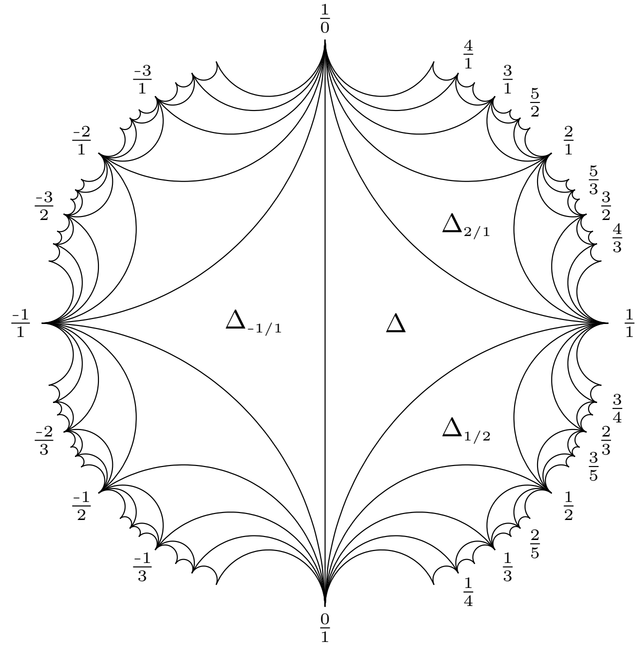

Figure 2: The Farey graph and Farey complex.

As seen in Figure 2, the edges in the Farey graph naturally form triangles. In fact, the vertices of any such triangle always have the form $\frac{p}{q}$, $\frac{r}{s}$ and $\frac{p+r}{q+s}$. For instance, $\frac{1}{0}, \frac{0}{1}$ and $\frac{1}{1}$, and also $\frac{1}{0}, \frac{1}{1}$ and $\frac{2}{1}$. There is an action of $\mathrm{SL}(2,\mathbb{Z})$ on the Farey graph defined by permuting the vertices using the rule:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \frac{p}{q} = \frac{ap+bq}{cp+dq}$$

It is easy to check that two vertices $\frac{p}{q}$ and $\frac{r}{s}$ are connected by an edge only if their images $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \cdot \frac{p}{q}$ and $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \cdot \frac{r}{s}$ are. Hence, this defines an action on the Farey graph and by extension on the Farey complex, which is the space we get by filling in the triangles in the Farey graph.

You have mostly likely seen this space and action before but under a different guise. Indeed, the Farey complex gives a tessellation of the hyperbolic plane by ideal triangles whose vertices in the upper half plane model are either rational or ∞. Moreover, the action described above is none other than the usual action of 2×2 matrices with real entries and positive determinant by fractional linear transformations of the upper half plane. The conformal maps:

$$f(z) = \frac{1-iz}{z-i} \text{ and } g(z) = \frac{1+iz}{z+i}$$

conjugate the two pictures. See Figure 3.

Now it is time to examine this action. Let Δ denote the triangle in the Farey complex with vertices $\frac{1}{0}$, $\frac{0}{1}$ and $\frac{1}{1}$. We record the key properties of the action in two claims.

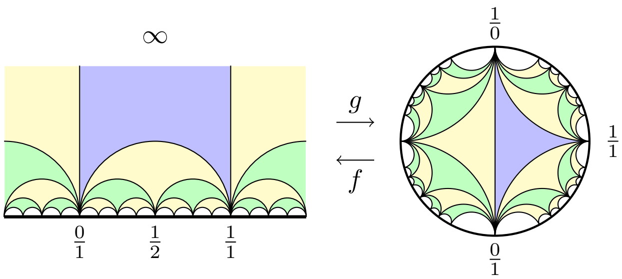

Figure 3: The Farey tessellation of the upper half plane by ideal triangles.

Claim 1. *For any triangle Δ' in the Farey complex, there is matrix $M \in \mathrm{SL}(2,\mathbb{Z})$ such that $M\Delta = \Delta'$.*

Indeed, suppose the vertices of Δ' are $\frac{p}{q}$, $\frac{r}{s}$ and $\frac{p+r}{q+s}$ where $ps - qr = 1$. Take $M = \left[\begin{smallmatrix} p & r \\ q & s \end{smallmatrix}\right]$ and observe that $M\Delta = \Delta'$.

Let $A = \left[\begin{smallmatrix} 0 & 1 \\ -1 & 1 \end{smallmatrix}\right]$. Notice that $A \cdot \frac{0}{1} = \frac{1}{1}$, $A \cdot \frac{1}{1} = \frac{1}{0}$ and $A \cdot \frac{1}{0} = \frac{0}{1}$ so that $A\Delta = \Delta$ and A acts on the triangle Δ by a rotation.

Claim 2. *If $M\Delta = \Delta$, then $M = A^k$ for some integer k.*

Indeed, if M fixes Δ, then it must cyclically permute the vertices $\frac{0}{1}$, $\frac{1}{1}$ and $\frac{1}{0}$. Hence $A^k M$ fixes the vertices $\frac{0}{1}$, $\frac{1}{1}$ and $\frac{1}{0}$ for some k. As the only conformal map that fixes three points is the identity, we see that $A^k M = \pm I$. The claim follows once we check that $A^3 = -I$.

Let $B = \left[\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right]$ and let $\Delta_{p/q}$ be the triangle that shares an edge with Δ and has the vertex $\frac{p}{q} \in \{\frac{-1}{1}, \frac{1}{2}, \frac{2}{1}\}$. These are labeled in Figure 2. We observe that $B\Delta = \Delta_{-1/1}$. As A rotates Δ, we also find that $AB\Delta = \Delta_{1/2}$ and $A^2B\Delta = \Delta_{2/1}$.

We are now in the position to show that $\mathrm{SL}(2,\mathbb{Z})$ is finitely generated by the matrices A and B. That is, any matrix in $\mathrm{SL}(2,\mathbb{Z})$ can be expressed as a product of A's and B's:

$$M = A^{m_1} B^{n_1} \cdots A^{m_k} B^{n_k}$$

for some integers m_j, n_j. Given $M \in \mathrm{SL}(2,\mathbb{Z})$ we want to consider a path in the Farey complex from Δ to $M\Delta$. What do we mean by path? Specifically, we mean a sequence of triangles $\Delta = \Delta_0, \ldots, \Delta_k = M\Delta$ where the triangles Δ_{j-1} and Δ_j share an edge.

Now we proceed via induction on the length of shortest path to $M\Delta$. Claim 2 handles the case that this length is 0. Next, using a path $\Delta = \Delta_0, \ldots, \Delta_k = M\Delta$ of minimal length we observe by Claim 1 and induction that $\Delta_{k-1} = M_0\Delta$ where M_0 can be expressed as product of A's and B's. Let's hit the whole picture with M_0^{-1}: the triangle $\Delta_{k-1} = M_0\Delta$ is sent to Δ and the triangle $\Delta_k = M\Delta$ is sent to an adjacent triangle, i.e., one of $\Delta_{-1/1}$, $\Delta_{1/2}$, or $\Delta_{2/1}$. Assuming for simplicity that $M_0^{-1}M\Delta = \Delta_{-1/1}$, which is equal to $B\Delta$, we find that $B^{-1}M_0^{-1}M\Delta = \Delta$. Claim 2 now shows that $B^{-1}M_0^{-1}M = A^k$ and hence $M = M_0BA^k$. Since M_0 can be expressed as a product of A's and B's, so can M, showing that $\mathrm{SL}(2,\mathbb{Z})$ is finitely generated.

A theorem: characterizing finite generation

What did we actually use to prove finite generation? The important topological property we used was the path-connectedness of the Farey complex so that we had a path from Δ to $M\Delta$ to apply induction on. The important dynamical property we used was the existence of a transitive tiling for which the stabilizer of a tile is finite and for which one tile meets only finitely many other tiles. These dynamical considerations naturally lead to the following definition.

Definition 1. An action of a group G on a metric space (X, d) by isometries is *geometric* if it satisfies the following two conditions:

1. (*cocompact*) there exists a compact set $K \subseteq X$ such that $\bigcup_{g \in G} gK = X$; and
2. (*properly discontinuous*) for any compact set $Y \subseteq X$, the set $\{g \in G \mid gY \cap Y \neq \emptyset\}$ is finite.

The requirement of a transitive tiling is captured by the cocompact condition. The properly discontinuous condition captures both requirements that the stabilizer of a tile is finite and that a tile meets only finitely many tiles.

Technical Sidenote (i.e. feel free to ignore): The actions of $\mathrm{SL}(2,\mathbb{Z})$ on the Farey complex and on the upper half plane are not geometric. For the Farey complex the action is cocompact, but a triangle intersects infinitely many other triangles at a vertex, so the action is not properly discontinuous. We got around this problem by only considering triangles that meet along an edge—there are only finitely many such. In the upper half plane the action is properly discontinuous, but the action is not cocompact. We can get around this by removing an equivariant collection of disjoint open disks tangent to the rational points. In either setting, the crucial point is that our notion of path ignores the vertices/ideal points. There is a geometric action lurking in the background here on the Farey tree that will be explored later.

Here are some examples of geometric actions.

1. The group $\mathbb{Z}^n$ acting by linearly independent translations on $\mathbb{R}^n$ equipped with the Euclidean metric.

2. More generally, any group of isometries of $\mathbb{R}^n$ equipped with the Euclidean metric that leaves a lattice $\Lambda \subset \mathbb{R}^n$ invariant and whose action on the lattice has finitely many orbits.

3. The fundamental group $\pi_1(X)$ of a compact Riemannian manifold X, possibly with boundary, acting by deck transformations on its universal cover $\widetilde{X}$ equipped with the pull-back metric.

Arguing as we did for $\mathrm{SL}(2,\mathbb{Z})$, we can prove the "if" direction of a geometric characterization of finite generation.

Theorem 1. *A group is finitely generated if and only if it acts geometrically on a path-connected metric space.*

For the "only if" direction, we need to introduce an important concept in geometric group theory: the Cayley graph.

A space for every group

For a finitely generated group G we need to produce a path-connected metric space that admits a geometric action by G. This is similar to what is required to prove Cayley's theorem from classical group theory: *Every group is isomorphic to a permutation group.* In the classical setting, we need to produce a set that admits a permutation action by our group. There is only one choice, the set is the group G and the action is left multiplication.

In our current setting, the idea is similar. The metric space is built on top of the group, the extra parts of the space come from a finite generating set. The result is called a Cayley graph. Here are the details.

Definition 2. Let G be a finitely generated group and let $S \subseteq G$ be a finite generating set. The *Cayley graph*, denoted $\Gamma(G,S)$, is the graph whose vertex set is G and where there is an edge joining vertices $h_1, h_2 \in G$ if $h_1^{-1}h_2 \in S$, i.e., $h_2 = h_1 s$ for some generator $s \in S$.

The group G acts on $\Gamma(G,S)$ by permuting the vertices via left multiplication. If vertices $h_1, h_2 \in G$ are adjacent, then so are the vertices gh_1, gh_2 as $(gh_1)^{-1}(gh_2) = h_1^{-1}h_2$ and so the permutation action on the vertices extends to the entire graph.

As S generates G, the Cayley graph $\Gamma(G,S)$ is path-connected. Figure 4 illustrates the path connecting the identity element of the group 1_G to the element $g = s_1 s_2 \cdots s_k$ where each s_j belongs to $S \cup S^{-1}$. The key point is that $s_1 \cdots s_j$ is adjacent to $s_1 \cdots s_{j+1}$.

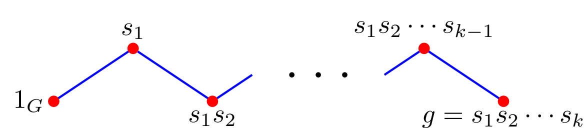

Figure 4: A path in the Cayley graph.

Here are some examples of Cayley graphs.

1. $\mathbb{Z}$ and $\mathbb{Z}^2$: For $\mathbb{Z}$, we can use $S = \{1\}$ and for $\mathbb{Z}^2$ we can use $S = \{\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right], \left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]\}$. These graphs are

pictured in Figure 5. Other generating sets are possible too, try drawing the graph $\Gamma(\mathbb{Z}, \{2,3\})$. You can find this graph in the essay by Margalit and Thomas [CM17, Office Hour 7].

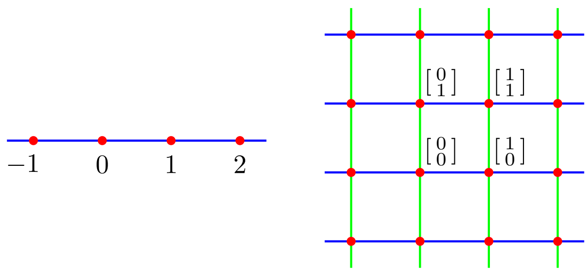

Figure 5: Cayley graphs for $\mathbb{Z}$ and $\mathbb{Z}^2$.

2. Sym(3): For the symmetric group on three elements, we can use the generating sets $S_1 = \{(1\ 2), (2\ 3)\}$ or $S_2 = \{(1\ 2), (1\ 2\ 3)\}$. These graphs are pictured in Figure 6 where elements in Sym(3) are listed using cycle notation.

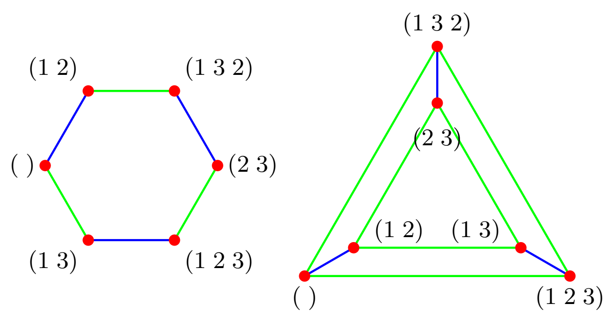

Figure 6: Cayley graphs for Sym(3).

3. F_2: For the free group of rank two, we can use a basis $S = \{a, b\}$. Recall that elements in F_2 are in one-to-one correspondence to words in the alphabet $\{a, a^{-1}, b, b^{-1}\}$ that are reduced in the sense that they do not contain aa^{-1}, $a^{-1}a$, bb^{-1} or $b^{-1}b$. For example, $a^2b^{-1}a^{-1}b$ and $b^2a^{-2}b^2$ represent elements in F_2. The group operation is concatenation followed by deletion of forbidden terms. As the reduced word representing an element is unique and as paths in the Cayley graphs read out a word representing an element as shown in Figure 4, there is a unique non-backtracking path from 1_{F_2} to any given element. Hence, the Cayley graph $\Gamma(F_2, \{a,b\})$ is a tree. A portion of this graph is pictured in Figure 7.

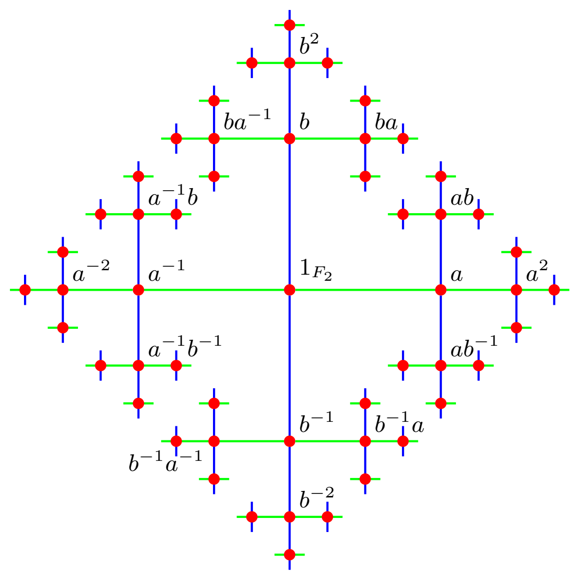

Figure 7: A Cayley graph for F_2.

There is a metric on the vertices of $\Gamma(G,S)$ defined as the minimum number of edges in an edge-path between a given pair of vertices. This metric can be extended to the points lying in edges by identifying (in an equivariant way) each edge with the unit interval $[0,1] \subset \mathbb{R}$. However for most applications in geometric group theory, having a metric only on the vertices suffices. The action of G on the Cayley graph $\Gamma(G,S)$ with this metric is by isometries.

The only item left to verify in Theorem 1 is that the action of G on $\Gamma(G,S)$ is geometric. We can easily check these in turn.

1. (cocompact) Let $K \subseteq \Gamma(G,S)$ be the union of the vertices $\{1_G\} \cup S$ together with the edges incident on 1_G and s for each $s \in S$. As S is finite, K is compact and clearly $\bigcup_{g \in G} gK = \Gamma(G,S)$.

2. (properly discontinuous) Suppose that $Y \subseteq \Gamma(G, S)$ is a finite subgraph and let n denote the number of vertices in Y. If $gY \cap Y \neq \emptyset$ then $gh_1 = h_2$ for a pair of vertices h_1, h_2 in Y and hence $g = h_2 h_1^{-1}$. Thus the cardinality of $\{g \in G \mid gY \cap Y \neq \emptyset\}$ is at most n^2.

Groups and spaces with negative curvature

In the previous section, we used a path-connected space and a geometric action to derive an algebraic consequence: finite generation. Path-connectivity is a fairly weak topological property, however the notion of a geometric action is quite restrictive. For instance, by proper discontinuity the subgroup fixing a given point must be finite. What can be gained from actions on spaces with more requirements on the topology and geometry, but perhaps fewer requirements on the dynamics of the action?

One geometric property that is particularly useful is the notion of negative curvature. We will look at two instances of negative curvature in geometric group theory: trees and δ–hyperbolic spaces.

Actions on trees

Negative curvature, say in the hyperbolic plane, influences the geometry of a space in several ways: uniqueness of geodesics, exponential growth in the volume of balls, and a uniform bound on the diameter of an inscribed circle to a triangle to name a few. To discuss the familiar notion of curvature from differential geometry, a space requires more structure than just an ordinary metric. Before discussing a notion of negative curvature expressed soley in terms of a distance function on an arbitrary set, let's consider an simple example of a metric space that has the properties listed above for the hyperbolic plane: a tree.

To see an example of the usefulness of group actions on trees, let's go back to the example of $\mathrm{SL}(2,\mathbb{Z})$ and think about its finite-order elements, i.e., matrices for which some positive power is equal to the identity. We can quickly compute that $A^3 = -I = B^2$, thus $A^6 = I$ and $B^4 = I$ and so A and B have finite order. Are there any others? There are obvious ones of course. Powers of A and powers of B clearly have finite order, as do their conjugates, CA^kC^{-1} and CB^kC^{-1} for any $k \in \mathbb{Z}$ and $C \in \mathrm{SL}(2,\mathbb{Z})$. But is that it? The answer to this last question is "yes" and we will see why using the action of $\mathrm{SL}(2,\mathbb{Z})$ on the Farey tree.

This tree is obtained from the Farey complex. Divide each triangle in the Farey complex into three quadrilaterals that meet pairwise along one leg of a tripod. Taken collectively these tripods form a tree, which is called the *Farey tree*. See Figure 8.

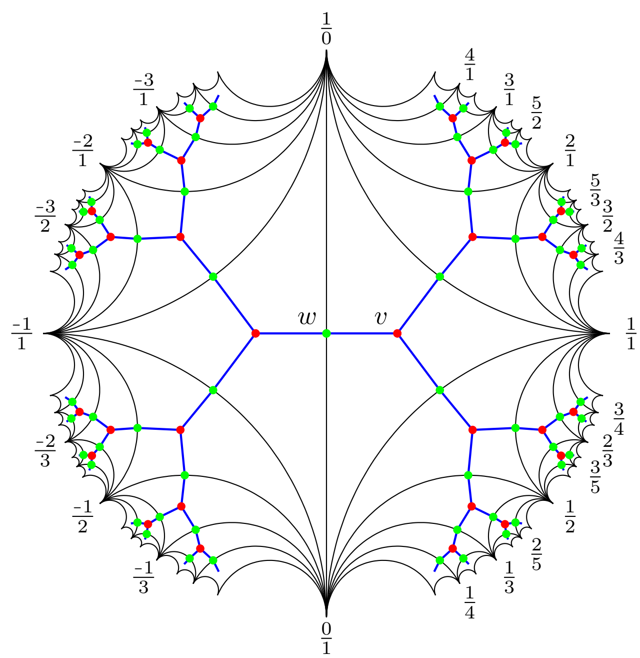

Figure 8: The Farey tree.

There are two types of vertices in the Farey tree: (red) degree three coming from the center of a triangle, and (green) degree two coming from an edge of a triangle. Let v denote the vertex that corresponds to the center of the triangle Δ and let w denote the vertex that corresponds to the edge in the Farey complex between $\frac{0}{1}$ and $\frac{1}{0}$. These are labeled in Figure 8.

From our study of the action of $\mathrm{SL}(2,\mathbb{Z})$ on the Farey complex, we conclude that every vertex in the Farey tree is a translate of v or w. This follows from Claim 1 and the fact that A cyclically permutes the edges of Δ and hence all of the vertices adjacent to v. Additionally, we can conclude from Claim 2 that the stabilizer of v is the cyclic subgroup of order 6 generated by A, and that the stabilizer of w is the cyclic subgroup of order 4 generated by B.

An important property of an action on a tree is the following claim.

Claim 3. *Suppose that a group G acts on a tree. If $g \in G$ has finite order, then g has a fixed point.*

The key fact here is that a finite set of points $x_1, \ldots, x_n$ in a tree has a unique *center*, i.e., a point c that minimizes the quantity

$$\max\{d(c, x_j) \mid j = 1, \ldots, n\}.$$

The center is easy to characterize. Suppose that x_1 and x_2 maximize $d(x_j, x_{j'})$ for $j, j' = 1, \ldots, n$. One can show that the center is the unique point c with $d(c, x_1) = d(c, x_2) = \frac{1}{2}d(x_1, x_2)$. Now fix a point x in the tree and let c be the center of the set $\mathcal{O} = \{x, gx, \ldots, g^{n-1}x\}$ where n is the order of g. Since the action is by isometries, we must have that gc is the center of the set $g\mathcal{O}$. But g permutes the points in $\mathcal{O}$, i.e., $g\mathcal{O} = \mathcal{O}$, and so $gc = c$.

Applying Claim 3 to the action of $\mathrm{SL}(2,\mathbb{Z})$ on the Farey tree, we see if $M \in \mathrm{SL}(2,\mathbb{Z})$ has finite order, then $Mx = x$ for some point x in this tree. If M fixes a point in the interior of an edge, then it must fix one of the incident vertices as well since these vertices have different degrees and cannot be interchanged by M. So we may assume that x is a vertex of the Farey tree. As every vertex is a translate of v or w, we have that $x = Cv$ or $x = Cw$ for some matrix $C \in \mathrm{SL}(2,\mathbb{Z})$. In the former, we observe that $(C^{-1}MC)v = C^{-1}Mx = C^{-1}x = v$ and so $M = CA^kC^{-1}$ for some $k \in \mathbb{Z}$. Similarly, in the latter, we conclude that $M = CB^kC^{-1}$ for some $k \in \mathbb{Z}$. Hence every finite-order element in $\mathrm{SL}(2,\mathbb{Z})$ is conjugate to a power of A or B. This is exactly what we desired to show.

The action of $\mathrm{SL}(2,\mathbb{Z})$ on the Farey tree is geometric. The argument we gave shows that if a group acts geometrically on a tree, then there are only finitely many conjugacy classes of finite-order elements. Indeed, by Claim 3 and since the action is cocompact, any finite order element is conjugate into one of finitely many stabilizer subgroups. Since the action is properly discontinuous, each of these subgroups is finite and so the result follows.

We can replace the assumption of proper discontinuity of the action with the assumption that each point stabilizer subgroup has finitely many conjugacy classes of finite-order elements and reach the same conclusion.

Theorem 2. *Suppose G acts cocompactly on a tree. If every point stabilizer has finitely many conjugacy classes of finite-order elements, then so does G.*

Theorem 2 illustrates a common paradigm in geometry group theory. If some property P holds for groups acting geometrically on a certain type of metric space, then the same should be true for a group G acting on this same type of metric space so long as certain subgroups (e.g., point stabilizers) have property P. In other words, we should be able to promote a property P from a collection of subgroups to the whole group G if we can find the appropriate space where these subgroups are the point stabilizers.

This idea suggests a useful strategy. Suppose you have some family of groups that fit into a hierarchy: $\mathcal{G}_0$, $\mathcal{G}_1$, $\mathcal{G}_2$, ... where the groups in $\mathcal{G}_0$ act geometrically on a certain type of metric space and the groups in $\mathcal{G}_k$ also act on this same type of metric space with point stabilizers belonging to $\mathcal{G}_{k-1}$. If we can verify the above paradigm for this type of metric space, this gives an inductive way to show that all the groups in this family have some particular property or structure. In the next section, we will mention an instance where this strategy has been particularly fruitful: the mapping class group of an orientable surface.

Actions on δ–hyperbolic spaces

Actions on trees are nice to work with, but they form a fairly restrictive class of groups. There are many interesting and natural groups in which every action on a tree has a global fixed point. For example, this is true for $\mathrm{SL}(n, \mathbb{Z})$ when $n \geq 3$. Surely, not much can be gained in general from actions with a global fixed point.

Gromov's influential essay [Gro87] introduced a notion of negative curvature that unifies essential properties of the hyperbolic plane, trees and small cancellation groups—a thoroughly studied class of groups explored in the latter half of the 20th century in which geometric notions and techniques were starting to gain traction. The idea behind Gromov's definition of a δ–hyperbolic space is to take one of the useful consequences of negative curvature from the hyperbolic plane and use it as a definition for a metric space. Gromov gave such a definition solely using a metric d on an arbitrary set X, but the most common formulation used—and one that applies to almost all the spaces one comes across in geometric group theory—requires a geodesic metric space, which is defined as follows. A *geodesic* in a metric space (X, d) is a function $p\colon Y \to X$ where Y is a connected subset of $\mathbb{R}$ such that $d(p(s), p(t)) = |t - s|$ for all $s, t \in Y$. A *geodesic metric space* is a metric space (X, d) such that for all $x, y \in X$, there is a geodesic $p\colon [0, L] \to X$ with $p(0) = x$ and $p(L) = y$. A connected graph, in particular the Cayley graph of a finitely generated group, is a geodesic metric space.

There are many equivalent formulations of a δ–hyperbolic metric space using geodesic triangles, divergence of geodesics, or nearest point projections to geodesics. We will state the most common formulation using geodesic triangles, which Gromov attributed to Rips. In the statement, $[a, b]$ represents any geodesic in X from a to b.

Definition 3. Let (X, d) be a geodesic metric space. A geodesic triangle $\Delta(a, b, c)$ is *δ–thin* if the δ–neighborhood of any two of the edges contains the third. That is, for all $x \in [a, c]$ there is an $x' \in [a, b] \cup [b, c]$ where $d(x, x') \leq \delta$. A *δ–hyperbolic* space is a geodesic metric space where every geodesic triangle is δ–thin.

The key point in the definition is that the same δ works for every geodesic triangle, no matter how long the sides are. See Figure 9.

Here are some examples of δ–hyperbolic spaces.

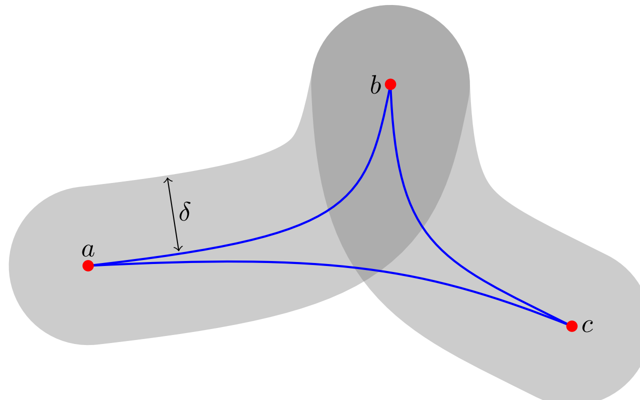

Figure 9: A δ–thin triangle.

1. A tree is 0–hyperbolic since every geodesic triangle is a tripod and so any side is contained in the union of the other two. See Figure 10. We think of thinner triangles indicating the space being more negatively curved—this is true for scalar curvature in Riemannian geometry—and so in this sense, trees are negatively curved in the extreme.

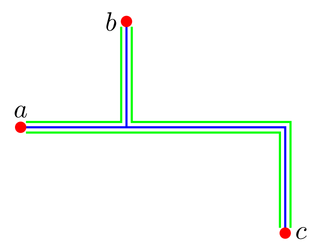

Figure 10: A typical geodesic triangle in a tree.

2. The hyperbolic plane is $\log(1 + \sqrt{2})$–hyperbolic. As every geodesic triangle is contained in an ideal triangle, we only have to compute δ for an ideal triangle, which is a fun exercise.
3. The Farey graph is 1–hyperbolic. This follows as the removal of any edge and its incident vertices disconnects the Farey graph.

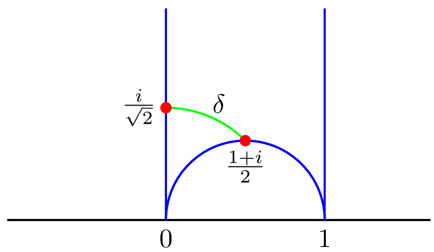

Figure 11: Ideal triangles in the the hyperbolic plane are $\log(1+\sqrt{2})$–thin.

For contrast, $\mathbb{R}^2$ with the Euclidean metric is not δ–hyperbolic for any δ. Indeed, the geodesic triangle with vertices $(0,0)$, $(n,0)$ and $(0,n)$ is δ–thin only for $\delta \geq n/2$. To see this, consider the point $(n/2, n/2)$.

The typical questions one may try to answer using actions on δ–hyperbolic spaces often fit into the following categories.

1. Algorithmic: When do two words in a generating set represent the same element or conjugate elements?

2. Local-to-global: Are paths in the Cayley graph that are locally geodesics globally geodesics as well?

3. Rigidity: If two groups have geometrically similar Cayley graphs, are the groups algebraically similar? Can we characterize homomorphisms to and from the group?

We will discuss in turn geometric actions and other types of actions on δ–hyperbolic spaces.

Geometric actions on δ–hyperbolic spaces

A metric space is *proper* if closed balls are compact. A group G is *hyperbolic* if it acts geometrically on a proper δ–hyperbolic space[2]. Free groups and fundamental groups of closed hyperbolic manifolds are hyperbolic groups. It is fair to ask how common hyperbolic groups are given that we started this section noticing that useful tree actions do not always exist. Gromov introduced a model of a "random finitely presented group" that includes a parameter $0 < d < 1$ called the "density" that controls the number of relators in terms of the number of generators [Gro93, Chapter 9]. When $d < 1/2$, Gromov showed that a random group is infinite and hyperbolic. (For those curious, when $d > 1/2$ a random group has at most two elements.) Thus, it is fair to say that hyperbolic groups are quite ubiquitous.

An equivalent definition of a hyperbolic group is that G is finitely generated and the Cayley graph $\Gamma(G,S)$ is δ–hyperbolic for some finite generating set $S \subseteq G$. Moreover, "some" in the previous sentence can be replaced with "every." Hyperbolic groups satisfy a long list of useful properties and besides Gromov's original essay, there are many comprehensive works focused on these groups. See for instance the notes edited by Short [ABC$^+$91], the chapters by Bridson and Haefliger [BH99, Chapters III.H and III.Γ], and the references within these works.

As hyperbolic groups are defined by a geometric condition (in several equivalent ways), from their inception researchers have wondered if there is an algebraic characterization. It is not too difficult to find algebraic obstructions. One of the first usually encountered involves the centralizer of an infinite-order element. If G is a hyperbolic group and $g \in G$ has infinite order, then $\langle g \rangle$, the cyclic subgroup generated by g, has finite index in $C_G(g)$, the centralizer of g. Recall, the centralizer of g is the subgroup of G consisting of elements $h \in G$ with $hg = gh$. The idea behind this fact nicely illustrates a typical geometric argument using the δ–thin triangle condition.

Suppose that $hg = gh$ and consider the four vertices 1_G, g^k, hg^k, and h in the Cayley graph $\Gamma(G,S)$ for a large k. The fact that $hg^k = g^k h$ implies that these four points lie on a rectangle. The horizontal sides are formed by a geodesic $[1_G, g^k]$ and its translate by h, the geodesic $[h, hg^k]$. To get the vertical sides, use a geodesic $[1_G, h]$ and its translate by g^k. The translate by g^k gives a geodesic from g^k to $g^k h$, but this latter point is exactly hg^k by the commutiv-

[2]In the literature, these groups are sometimes referred to as negatively curved, word hyperbolic or Gromov hyperbolic.

ity assumption. See Figure 12.

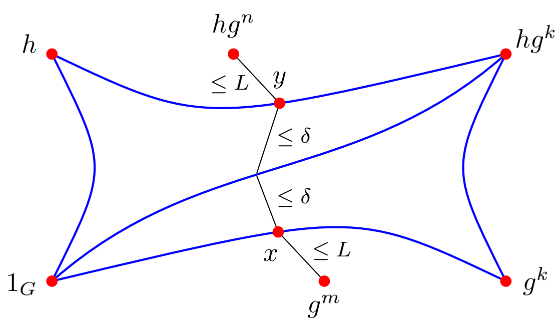

Figure 12: A commuting rectangle in $\Gamma(G,S)$.

There is a constant L so that any point on $[1_G, g^k]$ is within L of g^j for some $j = 0, \ldots, k$. Likewise, any point on $[h, hg^k]$ is within L of hg^j for some $j = 0, \ldots, k$. Now let x be the midpoint of the geodesic $[1_G, g^k]$. By considering the two geodesic triangles $\Delta(1_G, g^k, hg^k)$ and $\Delta(1_G, hg^k, h)$ pictured in Figure 12, we see that x is within 2δ of a point y that lies on one of other three sides of the rectangle. By choosing k large enough, we can ensure that y lies on the geodesic $[h, hg^k]$ as shown in Figure 12. We have $d(x, g^m) \leq L$ and $d(y, hg^n) \leq L$ for some $0 \leq m, n \leq k$ which gives

$$\begin{aligned} d(1_G, hg^{n-m}) &= d(g^m, hg^n) \\ &\leq d(g^m, x) + d(x, y) + d(y, hg^n) \\ &\leq 2L + 2\delta. \end{aligned}$$

Hence the coset $h\langle g\rangle \subseteq C_G(g)$ has an element whose distance from 1_G is at most $2L+2\delta$. As there are only finite many such elements and as distinct cosets are always disjoint, there are only finitely many cosets.

As a consequence, no subgroup of a hyperbolic group can be isomorphic to $\mathbb{Z}^2$. In several classes of geometrically defined groups, this turns out to be the only obstruction to hyperbolicity. For instance, this is true for the class of fundamental groups of closed 3–manifolds. In general, there is another algebraic obstruction to consider. Hyperbolic groups cannot contain a subgroup isomorphic to one of the Baumslag–Solitar groups:

$$BS(m, n) = \langle a, t \mid ta^m t^{-1} = a^n \rangle.$$

The notation here means that $BS(m, n)$ is generated by two elements a and t and the only relation they satisfy is that t conjugates a^m to a^n. This is a very interesting class of groups that includes $\mathbb{Z}^2$, which is $BS(1, 1)$, and the fundamental group of the Klein bottle, which is $BS(1, -1)$.

The reason hyperbolic groups cannot contain subgroups isomorphic to a Baumslag–Solitar group relies on the two facts that (1) for $k \geq 1$ the subgroup $\langle a^k, t^k \rangle \subseteq BS(m, n)$ is never free as $t^k a^{km^k} t^{-k} = a^{kn^k}$, whereas (2) for infinite-order elements $g, h \in G$ in a hyperbolic group, the subgroup $\langle g^k, h^k \rangle$ is free for some large k.

It was an open question until recently if this is essentially the only obstruction. Specifically, is a group G for which there exists a finite Eilenberg–MacLane space $K(G, 1)$ and that does not contain a subgroup isomorphic to $BS(m, n)$ necessarily hyperbolic? Brady gave counterexamples without the finiteness assumption [Bra99]. These examples were difficult to construct. They arise as subgroups of hyperbolic groups and hence do not have subgroups isomorphic to Baumslag–Solitar groups. The difficult part in the construction is showing these subgroups are not hyperbolic. Brady does so by showing they do not satisfy an algebraic finiteness condition, called F_2, known to be satisfied by hyperbolic groups. As the existence of a finite $K(G, 1)$ implies F_2 and also implies additional algebraic finiteness conditions satisfied by hyperbolic groups, a positive answer to the above question seemed plausible. Recently, Italiano, Martelli and Migliorini constructed a subgroup of a hyperbolic group that is not hyperbolic but does have a finite $K(G, 1)$, answering the above question in the negative [IMM]. The hyperbolic group they construct is a quotient of the fundamental group of a finite volume cusped hyperbolic 5–manifold.

Other actions on δ–hyperbolic spaces

There are many natural groups that contain subgroups isomorphic to $\mathbb{Z}^2$ and hence cannot be hyperbolic. Can we still use negative curvature to investigate these groups? Let's relax the condition of a geometric action, the requirement of a proper metric space, and consider an example of an important group in low-dimensional topology.

The *mapping class group* $\mathrm{MCG}(\Sigma)$ of an orientable surface Σ, possibly with boundary, is the the group of orientation preserving homeomorphisms of Σ modulo isotopy. That is, two homeomorphisms of Σ determine the same mapping class if one can be continuously deformed to the other so that every intermediate map along the way is also a homeomorphism. When Σ has non-empty boundary, the homeomorphisms and the isotopies need to be the identity on each boundary component. To simplify the discussion, we will assume that Σ is compact. This group appears in the study of 3–manifolds, algebraic geometry, cryptography, symplectic geometry, dynamics and configuration spaces. Using homeomorphisms supported on disjoint subsurfaces in Σ, it is easy to find subgroups isomorphic to $\mathbb{Z}^2$ in most mapping class groups. Thus $\mathrm{MCG}(\Sigma)$ is not hyperbolic in general.

The mapping class group acts on the *curve graph* $\mathcal{C}(\Sigma)$. A *simple closed curve* is an embedding of the circle $S^1 \to \Sigma$ that does not bound a disk nor an annulus in Σ (the latter only occurs when Σ has boundary). The curve graph is the graph whose vertex set is the set of isotopy classes of simple closed curves and two such $[c_0]$, $[c_1]$ are joined by an edge if they have disjoint representatives. In Figure 13 some curves on Σ are shown along with the corresponding subgraph of $\mathcal{C}(\Sigma)$. A mapping class $[f]$ acts on a vertex $[c]$ in the curve graph by sending the simple closed curve to its image: $[f] \cdot [c] = [f(c)]$. Homeomorphisms take disjoint curves to disjoint curves so this extends to an action on $\mathcal{C}(\Sigma)$ as well.

When the genus of Σ is equal to 1, i.e., when Σ is a torus $S^1 \times S^1$, any two non-isotopic simple closed curves necessarily intersect and so the above defition results in a graph with no edges. In this case the definition is altered slightly, $[c_0]$, $[c_1]$ are joined by an edge if they have representatives that intersect once. Let's take a closer look at this curve graph. Any simple closed curve on the torus is isotopic to one that winds p times around the first S^1 factor

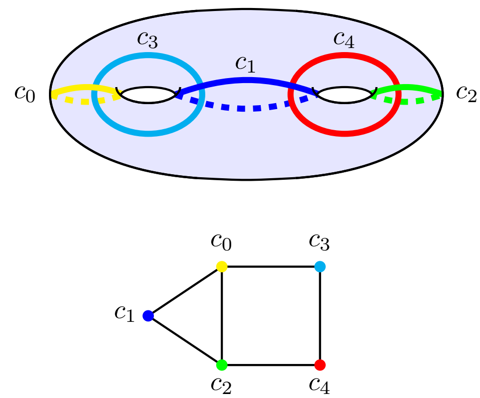

Figure 13: A portion of the curve graph for a genus 2 surface.

and q times around the second S^1 factor where p and q are relatively prime. As the orientation does not matter, we can assume that q is positive. That is, isotopy classes of simple closed curves on the torus are parameterized by the set of rational numbers $\frac{p}{q}$ along with an additional element $\frac{1}{0}$. Moreover, the number of times the simple closed curves $\frac{p}{q}$ and $\frac{r}{s}$ intersect is $|ps - qr|$.

Sound familiar? That's right, the curve graph of the torus is the Farey graph! In fact, the mapping class group of the torus is isomorphic to $\mathrm{SL}(2,\mathbb{Z})$ and the two actions are the same. The action of $\mathrm{SL}(2,\mathbb{Z})$ on the Farey graph illustrates some of the essential properties of $\mathcal{C}(\Sigma)$ and the action of $\mathrm{MCG}(\Sigma)$ on $\mathcal{C}(\Sigma)$.

First, as we observed for the Farey graph, the curve graph is δ–hyperbolic. This amazing fact was proved by Masur and Minsky [MM99] and has been reproved a number of times since. (The best estimate on δ was given by Hensel, Przytycki and Webb who showed that $\delta \leq 17$ [HPW15].) It is impossible to overstate the influence of this result on the study of the mapping class group, the geometry of 3–manifolds and geometric group theory in general.

Second, the action is not properly discontinuous. Indeed, the vertex stabilizers are infinite. This not a bug though, but a feature! Homeomorphisms that fix a simple closed curve c are actually homeomorphisms of the surface obtained by cutting open along c. Thus the stabilizer of a vertex in $\mathcal{C}(\Sigma)$ is the mapping class group of a surface Σ', possibly disconnected, whose components are simpler in the sense that the genus or the number of boundary components is fewer than that of Σ.

Taken together, this setting fits in to the hierarchy strategy mentioned after Theorem 2 and the number of applications are extremely numerous. I will mention one here that ties back to the beginning of the article: finite generation. Using the facts that the curve graphs are path-connected and that the stabilizers of vertices in the curve graph are, by induction, finitely generated, it can be shown that the mapping class group of any orientable surface is also finitely generated. That is, we promote finite generation from the stabilizers to whole group using the fact that the space is path-connected. The base case for the induction is when the surface has genus 1 and a single boundary component. The mapping class group of this surface is $\mathrm{SL}(2,\mathbb{Z})$—the group we started our journey with! This strategy was orginally employed by Dehn and he found a specific generating set for the mapping class group analogous to elementary matrices. For complete details, and much more on mapping class groups, see the text by Farb and Margalit [FM12].

Other useful properties and features of the mapping class group acting on the curve graph have been identified, isolated and applied to the study of other groups. These include the notion of a *WPD element* by Bestvina and Fujiwara [BF02], the notion of a *projection complex* by Bestvina, Bromberg and Fujiwara [BBF15], the notion of an *acylindrical action* by Osin [Osi16] and the notion of a *hierarchically hyperbolic group/space* by Behrstock, Hagen and Sisto [BHS17]. The common element of each of these new tools is to exploit negative curvature in certain directions of the group. As in the case of the mapping class group acting on the curve graph, the applications to a variety of classes of groups have been numerous.

Conclusion and further reading

I hope the above gives you an idea of how the topology and geometry of a space that a group acts on can influence its algebraic properties and structure. Geometric group theory is a growing field. This is in part due to the large number of questions the field generates regarding the geometry of finitely generated groups, but the field has also seen an increase in interest as a result of its applications to other areas of mathematics. A striking example of this is the recent resolution of the Virtual Haken Conjecture in hyperbolic geometry. This was proved by Agol [Ago13] using tools from geometric group theory created by Scott, Sageev, Wise and others. See the survey article by Bestvina [Bes14] for an excellent overview of this connection.

The concept of δ–hyperbolicity is but one aspect of geometric group theory. There are areas of geometric group theory invoking tools from algebra (algebraic geometry, homological algebra) analysis (L^p–spaces, C^* and von Neumann algebras), dynamics (entropy, topological Markov chains), geometry (isoperimetric functions, Lie theory) and topology (dimension, fractals). Below is a selection of books on geometric group theory for those curious to learn more, listed by publication date.

1. *Metric Spaces of Non-positive Curvature* by Martin Bridson and André Haefliger [BH99]: Comprehensive reference text focusing on various notions of non-positive curvature in metric spaces and groups.

2. *Topics in Geometric Group Theory* by Pierre de la Harpe [dlH00]: An introduction to groups as geometric objects including a multitude of examples and a broad investigation on the notion of growth in groups.

3. *A Course in Geometric Group Theory* by Brian Bowditch [Bow06]: Introductory text based on a course taught by the author with an in depth treatment of hyperbolic groups. The audience is advanced undergraduates and beginning graduate students.

4. *Office Hours with a Geometric Group Theorist* edited by Matt Clay and Dan Margalit [CM17]: A collection of essays written by researchers on select topics in geometric group and central examples such as Coxeter groups and braid groups targeted to advanced undergraduates and beginning graduate students.

5. *Geometric Group Theory* by Clara Löh [Löh17]: Introductory text on geometric group theory targeted to advanced undergraduates and beginning graduate students. Fundamental topics such as quasi-isometry, boundaries and amenable groups are discussed.

6. *Geometric Group Theory* by Cornelia Druţu and Michael Kapovich [DK18]: Comprehensive text containing proofs of several fundamental results in geometric group theory including the Tits alternative and Gromov's theorem on polynomial growth. The audience is advanced graduate students and researchers in the field.

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