DEFORMATION SPACES OF G–TREES AND AUTOMORPHISMS OF BAUMSLAG–SOLITAR GROUPS

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ABSTRACT. We construct an invariant deformation retract of a deformation space of G–trees. We show that this complex is finite dimensional in certain cases and provide an example that is not finite dimensional. Using this complex we compute the automorphism group of the classical non-solvable Baumslag–Solitar groups $BS(p,q)$. The most interesting case is when p properly divides q. Collins and Levin computed a presentation for $\mathrm{Aut}(BS(p,q))$ in this case using algebraic methods. Our computation uses Bass–Serre theory to derive these presentations. Additionally, we provide a geometric argument showing $\mathrm{Out}(BS(p,q))$ is not finitely generated when p properly divides q.

INTRODUCTION

Baumslag–Solitar groups have the following standard presentations:

$$BS(p,q) = \langle x, t \mid tx^p t^{-1} = x^q \rangle. \tag{1}$$

When p properly divides q there are infinitely many similar presentations for $BS(p,q)$ which highlights additional symmetries. These groups were first studied by Baumslag and Solitar as some examples of non-Hopfian groups [4]. Our interest is in the automorphism and outer automorphism groups, $\mathrm{Aut}(BS(p,q))$ and $\mathrm{Out}(BS(p,q))$ respectively, of the non-solvable Baumslag–Solitar groups. For non-solvable Baumslag–Solitar groups neither $|p|$ nor $|q|$ equals 1. By interchanging $t \leftrightarrow t^{-1}$, we can always assume that $|q| \geq p > 1$.

Presentations for these automorphism groups are known. The first result was by Collins, who gave a finite presentation for $\mathrm{Aut}(BS(p,q))$ when p and q are relatively prime [12]. This result was extended by Gilbert, Howie, Metaftsis and Raptis to the cases when p does not divide q or when $p = |q|$ [19]. Collins and Levin had earlier studied the most interesting cases, which is when when p properly divides q [13]. In this case $\mathrm{Aut}(BS(p,q))$ is not finitely generated. A summary of these results appears in Section 3. Although we do not consider the solvable case, we note that Collins found a finite presentation for the automorphism group $\mathrm{Aut}(BS(1,q))$ that depends on the prime factorization of q [12].

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One of the purposes of this paper is to a give unified approach to the computation of these automorphism groups. To this end, we construct a tree $X_{p,q}$ on which $\mathrm{Out}(BS(p,q))$ acts. In the cases Gilbert et al. considered, the tree $X_{p,q}$ is a single point. In the more interesting cases, when p properly divides q, this tree is nontrivial. The arguments from Gilbert et al. are used to compute the vertex stabilizers of the tree $X_{p,q}$. Using Bass–Serre theory in the case where p properly divides q we recover the presentations of $\mathrm{Aut}(BS(p,q))$ and $\mathrm{Out}(BS(p,q))$ (Theorem 4.4) originally found by Collins and Levin [13]. Prior to finding these presentations, we present a simple geometric argument showing that $\mathrm{Out}(BS(p,q))$ (and hence $\mathrm{Aut}(BS(p,q))$ is not finitely generated if p properly divides q (Theorem 4.3).

The key construction is an invariant deformation retract within deformation space of G–trees. This is the second purpose of this paper. The definition of a deformation space (of G–trees) appears in the next section, but loosely speaking a deformation space $\mathcal{D}$ is a moduli space of certain tree actions for a finitely generated group. These spaces were introduced by Forester [16]. Culler and Vogtmann's Outer space is a celebrated example of a deformation space [15]. Following intuition from Outer space, we define a deformation retract $W \subset \mathcal{D}$ of a general deformation space. When the deformation space is Outer space, the deformation retract W coincides with the spine of reduced Outer space. In some cases we can prove that W is finite dimensional (Theorem 1.18) and we provide an example of a deformation space $\mathcal{D}$ for which the deformation retract W is not finite dimensional (Example 2.2). We note that Guirardel and Levitt have defined a similar deformation retract within a non-ascending deformation space [20].

A non-solvable Baumslag–Solitar group $BS(p,q)$ has a canonical deformation space $\mathcal{D}_{p,q}$. This deformation space is invariant under the action of $\mathrm{Out}(BS(p,q))$. The deformation retract of $\mathcal{D}_{p,q}$ mentioned in the preceding paragraph is denoted $W_{p,q}$. We use this deformation space to construct the tree $X_{p,q}$. If p does not divide q or $p = |q|$, then $W_{p,q}$ is a single point and we take $X_{p,q} = W_{p,q}$. If $|q|/p$ is prime, then the deformation retract $W_{p,q}$ is a tree and we set $X_{p,q} = W_{p,q}$. In the other cases the complex $W_{p,q}$ is not a tree. Using our description of $W_{p,q}$ when $|q|/p$ is prime, we define an $\mathrm{Out}(BS(p,q))$–invariant subcomplex of $W_{p,q}$ and prove that this subcomplex is a tree (Theorem 3.9). In this case we take $X_{p,q}$ to be this tree.

We are hopeful that these techniques can extend to computing the finiteness properties of other outer automorphism groups using deformation spaces.

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1. Deformation spaces of G–trees

1.1. **Definitions.** In this section we define the preliminary notions essential to the following.

For a graph Γ, we denote by $V(\Gamma)$ the set of vertices of Γ and by $E(\Gamma)$ the set of oriented edges of Γ. For an edge $e \in E(\Gamma)$, $\bar{e}$ denotes the edge e with the opposite orientation. The attaching maps are $o, t\colon E(\Gamma) \to V(\Gamma)$ (originating (initial) and terminating vertices). For a vertex $v \in V(\Gamma)$, let $E_o(v) = \{e \in E(\Gamma) \mid o(e) = v\}$. An *edge path* $\gamma = (e_0, \ldots, e_n)$ is a set of oriented edges such that $t(e_s) = o(e_{s+1})$ for $s = 0, \ldots, n-1$. A *circuit* is an edge path that is homeomorphic to a circle and a *loop* is a circuit consisting of a single edge.

Let G be a finitely generated group. Later we will specialize to Baumslag–Solitar groups, but the results and constructions in this first section apply to a general finitely generated group.

A *G–tree* T is a simplicial tree that admits an action of G by simplicial automorphisms, i.e., by maps $f\colon T \to T$ that are bijections on the sets of edges and vertices and such that $f(o(e)) = o(f(e))$ for all $e \in E(T)$. Actions are always assumed to be without inversions, i.e., $ge \neq \bar{e}$ for all $g \in G, e \in E(T)$. Two G–trees are equivalent if there is a G–equivariant simplicial isomorphism between them. A *metric G–tree* is a G–tree with a metric such that the action of G is by isometries. As such we will consider two metric G–trees equivalent if there is a G–equivariant isometry between them. In either case, we will always assume that the the G–tree T is *minimal* (no invariant subtree), thus T/G is a finite graph.

For a metric G–tree T, the *length function*, $l_T\colon G \to \mathbb{R}$, is defined as $l_T(g) = \inf_{x \in T} d_T(x, gx)$. It is well-known that this infimum is achieved. When we speak of a length function for a simplicial G–tree T, we mean the length function when we consider T as a metric G–tree where all edges are assigned length one. The *characteristic subtree* for an element $g \in G$ is $T_g = \{x \in T \mid d_T(x, gx) = l_T(g)\}$. If $l_T(g) > 0$, then T_g is isometric to $\mathbb{R}$ and g acts on T_g as a translation by $l_T(g)$. In this case, g is called *hyperbolic* and T_g is called the *axis* of g. Culler and Morgan proved that irreducible G–trees are uniquely determined by their length functions [14]. A G–tree is *irreducible* if G does not fix an end of T nor a pair of ends. An equivalent condition is that there are two hyperbolic elements whose axes are either disjoint or intersect in a compact set [14].

For a given G–tree T, a subgroup $H \subseteq G$ is *elliptic* if H fixes a point in T. There are two ways to modify a G–tree, called *collapse* and *expansion*, that do not change the set of elliptic subgroups. These two moves correspond to the graph of groups isomorphism $A *_C C \cong A$. A finite sequence of these moves is called an *elementary deformation*. Conversely, Forester proved that any two G–trees with the same set of elliptic subgroups are related by an elementary deformation [16]. The definitions of collapse and expansion are as follows.

Definition 1.1 (*Collapse*)**.** Let T be a G–tree and suppose there is a vertex $v \in V(T)$ and an edge $e \in E_o(v)$ such that $G_e = G_v$ and $o(e)$ is not G–equivalent to $t(e)$, i.e., the image of e in T/G is not a loop. We define a new G–tree T_e by removing the edges Ge, then for all edges $f \in E(T)$ such that $o(f) = t(ge)$ for some $g \in G$ redefine $o(f) = gv$. This is a *collapse* move. Such edges e are called *collapsible*.

If a G–tree does not admit a collapse move, it is called *reduced*.

Definition 1.2 (*Expansion*)**.** Let T be a G–tree and $v \in V(T)$. Given a subgroup $H \subseteq G_v$ and an H–invariant set of edges $S \subseteq E_o(v)$ such that $G_e \subseteq H$ for all $e \in S$ we can define a new G–tree $T^{H,S}$ by adding a new edge i via $o(i) = v$ and redefining $o(e) = t(i)$ for $e \in S$, then repeating for all cosets $gH \in G/H$ using with the subgroup $gHg^{-1} \subseteq G_{gv}$ and the set of edges $gS \subseteq E_o(gv)$. This is an *expansion* move.

The stabilizer of the new edge i and the vertex $t(i)$ is H.

There are three special elementary deformations that we use frequently: *induction*, *slide* and $\mathscr{A}^{\pm 1}$. Pictures for these moves in the general setting appear in [11]. In Section 2 these moves are shown for the canonical deformation space associated to a generalized Baumslag–Solitar group.

Definition 1.3 (*Induction*)**.** Let T be a G–tree and suppose that at the vertex $v \in V(T)$ there are two edges $e, e' \in E_o(v)$ such that $G_e = G_v$, $G_{e'} \neq G_v$ and $e' \in G\bar{e}$, i.e., e and e' project to the same loop in T/G with opposite orientations. The composition of the expansion using any subgroup $G_{e'} \subseteq H \subsetneq G_v$ and the set $S = He'$ followed by the collapse of e is called an *induction*. The inverse of an induction is also called an induction.

The loop in T/G created by the image of e in the above definition is called an *ascending loop*, i.e. a loop in T/G in which one of the attaching maps is an isomorphism and the other is a proper inclusion. An induction move changes the stabilizer of the vertex v to H.

Definition 1.4 (*Slide*)**.** Let T be a G–tree and suppose that at the vertex $v \in V(T)$ there are two edges $e, f \in E_o(v)$ such that $G_f \subseteq G_e$ and f is not G–equivalent to e or $\bar{e}$. We can perform an expansion using the subgroup $H = G_e$ and set $S = \{e\} \cup G_e f$. Then in the expanded G–tree, the edge e is collapsible. The composition of this expansion followed by the collapse of e is called a *slide*.

A slide can also be thought of as removing the edge f at $o(f)$ and reattaching it via $o(f) = t(e)$, then repeating equivariantly throughout T. Topologically, we are folding the first half of f across e.

Definition 1.5 ($\mathscr{A}^{\pm 1}$*–move*)**.** Let T be a G–tree and suppose that at the vertex $v \in V(T)$ there are edges $e, e', f \in E_o(v)$ where e, e' satisfy the hypotheses of an induction move and $G_{e'} \subseteq G_f$. Further suppose that there are exactly three G_v–orbits

in $E_o(v)$ and the edges e, e' and f are in distinct orbits. Then there is an induction move after which $G_v = G_f$. The composition of this induction move followed by the collapse of the edge f is called an $\mathscr{A}^{-1}$–move. The inverse is called an $\mathscr{A}$–move.

An $\mathscr{A}$–move creates an ascending loop and an $\mathscr{A}^{-1}$–move removes an ascending loop.

Deformation spaces of G–trees were introduced by Forester [16]. Given a G–tree T, let $\mathcal{X}$ be the set of all metric G–trees that define the same set of elliptic subgroups as T. This set of metric G-trees is called an *unnormalized deformation space of G–trees*. There is an action of $\mathbb{R}^+$ on $\mathcal{X}$ by scaling the metric on a given metric G–tree. The quotient $\mathcal{D}$, is called a *deformation space of G–trees*, or sometimes just a *deformation space*. By Forester's deformation theorem, disregarding the metric on any two projectivized G–trees $T, T' \in \mathcal{D}$, there is an elementary deformation transforming T to T'.

The importance of the three special moves described above is given by the following theorem.

Theorem 1.6 ([11]). *In a deformation space of G–trees, any two reduced trees are related by a finite sequence of slides, inductions and $\mathscr{A}^{\pm 1}$–moves, with all intermediate trees reduced.*

Deformation spaces can be topologized in several ways: *axes topology, equivariant Gromov–Hausdorff topology* or *weak topology*. In the case of irreducible G–trees, the axes topology and the equivariant Gromov–Hausdorff topology are the same [26]. In addition for locally finite G–trees, the equivariant Gromov–Hausdorff topology and the weak topology are the same [20]. This is not true in general however, for an example see [25]. The topology we work with is the weak topology; for definitions of the other two see [8]. A projectivized G–tree $T \in \mathcal{D}$ determines an open simplex by equivariantly changing the lengths of the edges of T while holding the sum of the lengths of the edges in T/G constant. This open simplex has dimension one less than the number of edges of T/G. The faces in the closure of this open simplex are found by collapsing subsets of collapsible edges in T.

Culler and Vogtmann's Outer space is an example of a deformation space of G–trees [15]. In this case, G is a finitely generated free group of rank n at least 2, F_n, and the only elliptic subgroup is the trivial group. In other words, all of the actions are free. Another example is the complex $K_0(G)$ defined by McCullough and Miller [25]. Here, G is an arbitrary finitely generated group and the elliptic subgroups are the free product factors in a Grusko decomposition for G. Guirardel and Levitt describe a different complex for a free product decomposition of a finitely generated group G [21]. Their complex is the deformation space where the elliptic subgroups are the free product factors in a Grusko decomposition for G that are not infinite cyclic.

The deformation space $\mathcal{D}$ is acted upon by some subgroup $\mathrm{Out}(G)_{\mathcal{D}} \subseteq \mathrm{Out}(G)$. This is the subgroup that preserves the set of conjugacy classes of the elliptic subgroups associated to $\mathcal{D}$. In the case of Culler and Vogtmann's Outer space, this is the entire group $\mathrm{Out}(F_n)$. Likewise, the complex defined by Guirardel and Levitt is invariant under $\mathrm{Out}(G)$. For the McCullough–Miller complex $K_0(G)$, $\mathrm{Out}_{\mathcal{D}}(G) = \Sigma\mathrm{Out}(G)$, the subgroup of *symmetric outer automorphisms*. These examples are contractible and have been used for computing some of the finiteness properties of $\mathrm{Out}_{\mathcal{D}}(G)$, see individual references.

Under some mild hypotheses, we [8] and independently Guirardel and Levitt [20] have shown that deformation spaces are contractible. Both our proof and the proof of Guirardel and Levitt use Skora's method of continuous folding [30].

Theorem 1.7. *For a finitely generated group G, any irreducible deformation space that contains a G–tree with finitely generated vertex stabilizers is contractible.*

A deformation space is *irreducible* if all G–trees (equivalently a single G–tree) in $\mathcal{D}$ are (is) irreducible. In [8], the above theorem is only shown for the equivariant Gromov–Hausdorff topology (equivalently axes topology). Guirardel and Levitt show this as well as showing that the contraction is continuous in the weak topology. In addition, they replace irreducible with a weaker hypothesis.

1.2. **A deformation retract of $\mathcal{D}$.** Culler and Vogtmann's Outer space deformation retracts to a subcomplex called *reduced Outer space*. This is the subcomplex of projectivized free metric F_n–trees T such that the quotient graph T/F_n does not have any separating edges. This deformation retract is obtained by equivariantly shrinking the edges that project to separating edges in the quotient graphs T/F_n. There is a further deformation retraction of reduced Outer space to a spine. We will describe a similar deformation retract of a deformation space $\mathcal{D}$. For general deformation spaces, it is easier to deformation retract to the spine first, then define an additional deformation retraction.

Let K be the spine of $\mathcal{D}$. Specifically, let $\mathcal{OS}(\mathcal{D})$ be the poset of open simplices of $\mathcal{D}$ where $\sigma \leq \sigma'$ if σ is a face in the closure of σ'. The spine K is defined as the geometric realization of the poset $\mathcal{OS}(\mathcal{D})$. The spine K is an $\mathrm{Out}(G)_{\mathcal{D}}$–invariant deformation retraction of the deformation space $\mathcal{D}$ (with the weak topology). Therefore, $\mathrm{Out}(G)_{\mathcal{D}}$ acts on K, which is a contractible simplicial complex.

The spine K has an alternative combinatorial description in terms of the G–trees appearing in $\mathcal{D}$. The poset $\mathcal{OS}(\mathcal{D})$ is isomorphic to the poset $\mathcal{C}ol(\mathcal{D})$ of G–trees in $\mathcal{D}$ (thought of only as simplicial trees) where $T' \geq T$ if T' collapses to T. We say that T' *collapses to* T if there is a G–equivariant map $T' \to T$ that is a composition of finitely many collapse moves. Thus K can also be viewed as the geometric realization of the poset $\mathcal{C}ol(\mathcal{D})$. Vertices of K are G–trees in $\mathcal{D}$; higher dimensional simplices correspond to collapse sequences of such G–trees. To ease notation, we will use K

to denote the set of vertices of K. For future reference, we remark that no G–tree $T \in K$ contains a *subdivision vertex*, i.e. a vertex $v \in V(T)$ with $E_o(v) = \{e, f\}$ and $G_e = G_v = G_f$.

For a G–tree $T \in K$ define the following sets:

$$\begin{aligned} \mathrm{col}(T) &= \{T' \in K \mid T \geq T'\}, \\ \mathrm{red}(T) &= \{T' \in \mathrm{col}(T) \mid T' \text{ is reduced}\}, \text{ and} \\ \mathcal{W}(T) &= \{T' \in \mathrm{col}(T) \mid \mathrm{red}(T) = \mathrm{red}(T')\}. \end{aligned}$$

In the setting of Culler and Vogtmann's Outer space, $\mathrm{red}(T)$ is the set of F_n–trees found by collapsing some maximal forest of T/F_n and $\mathcal{W}(T)$ is the set of F_n–trees found by collapsing separating edges in T/F_n. The following lemma shows that the above sets can be realized as certain subsets of collapsible edges in T/G.

Lemma 1.8. *If T is an irreducible G–tree with collapsible edges e and f where $f \notin Ge \cup G\bar{e}$, then $T_e \neq T_f$. Also, if e is collapsible in T_f and f is collapsible in T_e, then $(T_e)_f = (T_f)_e$.*

Proof. If there is a hyperbolic element $g \in G$ whose axis projects down to a closed path in T/G that crosses the image of e more than it crosses the image of f then $l_{T_e}(g) < l_{T_f}(g)$. Thus as irreducible G–trees are determined by their length functions, $T_e \neq T_f$. It is easy to find such an element $g \in G$ by looking at edge paths (in the graph of groups sense) in T/G, see [1, 29].

By looking at length functions again, the second part of the lemma is obvious. □

The following definition appears in [20].

Definition 1.9. An edge $e \in E(T)$ is called *surviving* it there is a G–tree $T' \in \mathrm{red}(T)$ such that e is not collapsed in $T \to T'$. If an edge is not a surviving edge, it is called *non-surviving*.

Lemma 1.10. *$\mathcal{W}(T)$ has a unique minimal element, $T_{\mathcal{W}}$. This G–tree is characterized as the G–tree obtained from T by collapsing the non-surviving edges. Further, if $T \geq T'$ then $T_{\mathcal{W}} \geq T'_{\mathcal{W}}$.*

Proof. As $\mathcal{W}(T)$ is a finite set, minimal elements exist in $\mathcal{W}(T)$. Let T_0 be a minimal element. Then every non-surviving edge in T must be collapsed in $T \to T'_0$ since T_0 is minimal. Also, no surviving edge could be collapsed in $T \to T_0$ as $\mathrm{red}(T) = \mathrm{red}(T_0)$. Hence any minimal element T_0 is found by collapsing the non-surviving edges. By Lemma 1.8, this completely determines T_0.

Finally notice that if $T \geq T'$ then any non-surviving edge in T is either collapsed in $T \to T'$ or it is non-surviving for T'. This is true since if the edge is surviving for T', then it must also be surviving for T. Thus $T_{\mathcal{W}} \geq T'_{\mathcal{W}}$. □

Define $h\colon \mathcal{C}ol(\mathcal{D}) \to \mathcal{C}ol(\mathcal{D})$ by $h(T) = T_{\mathcal{W}}$. The above lemma shows that h is a well-defined poset map. Notice that $h(T) \leq T$. The following lemma shows that h defines a deformation retract of K, the geometric realization of $\mathcal{C}ol(\mathcal{D})$.

Lemma 1.11 (Quillen's Poset Lemma [28])**.** *Let X be a poset and $f\colon X \to X$ be a poset map (i.e. $x \leq x'$ implies $f(x) \leq f(x')$ for all $x, x' \in X$) with the property that $f(x) \leq x$ for all $x \in X$ (or $f(x) \geq x$ for all $x \in X$). Then the geometric realization of $f(X)$ is a deformation retract of the geometric realization of X.*

Define W as the geometric realization of the poset $h(\mathcal{C}ol(\mathcal{D}))$. Therefore, by Quillen's Poset Lemma, W is a $\mathrm{Out}(G)_{\mathcal{D}}$–invariant deformation retract of $\mathcal{D}$. In particular, W is contractible. For Culler and Vogtmann's Outer space, W is the spine of reduced outer space. We will explicitly show the deformation retraction $h\colon K \to W$ in Example 2.1 for a star in a deformation space for the Baumslag–Solitar group $BS(2,4)$. If $\mathcal{D}$ is a non-ascending deformation space, then W is the spine of the deformation retract defined by Guirardel and Levitt [20]. A deformation space $\mathcal{D}$ is called *non-ascending* if for all $T \in \mathcal{D}$, the quotient graph of groups T/G does not contain an ascending loop. As for K, we will use W to denote the set of vertices of W.

1.3. **G–trees in W.** In this section we determine when a G–tree in $\mathcal{D}$ represents a vertex in W. The following definition generalizes the definition of *shelter* in [20].

Definition 1.12. Let T be a G–tree, $\gamma = (e_0, \ldots, e_n) \subseteq T$ an edge path in T and $\widehat{\gamma}$ the image of γ in T/G. We say γ (or $\widehat{\gamma}$) is a *shelter* if either:

(S1) $\widehat{\gamma}$ is a topological segment, $G_{o(e_0)} \neq G_{e_0}$, $G_{e_n} \neq G_{t(e_n)}$ and $G_{e_s} = G_{t(e_s)} = G_{e_{s+1}}$ for $s = 0, \ldots, n-1$;
(S2) $\widehat{\gamma}$ is a circuit and $G_{e_s} = G_{t(e_s)} = G_{e_{s+1}}$ for $s = 0, \ldots, n-1$;
(S3) $\widehat{\gamma}$ is a circuit and $G_{o(e_s)} = G_{e_s}$ for $s = 0, \ldots, n$.

We refer to the labels S1, S2 or S3 as the *type* of the shelter. See Figure 1.

The following proposition generalizes Corollary 7.5(1) in [20].

Proposition 1.13. *An edge is surviving if and only if it is contained in a shelter.*

Before we prove this, we prove a simple lemma about how stabilizers can change after collapse moves.

Lemma 1.14. *Let f be a collapsible edge in T. Let $o(f) = v, t(f) = w$ and denote the image of f in T_f by z. Then for $e \in E(T)$, $e \notin Gf \cup G\bar{f}$ with $o(e) = v$, $G_e \neq G_z$ if and only if $G_e \neq G_v$ or $G_f \neq G_w$.*

Proof. As f is collapsible, $G_v \subseteq G_w$ or $G_w \subseteq G_v$, in either case G_z is the union of the two subgroups. Therefore if $G_e \subsetneq G_v$, then $G_e \neq G_z$. If $G_f \subsetneq G_w$, then as f is collapsible $G_f = G_v$. Therefore $G_e \subseteq G_v = G_f \subsetneq G_w = G_z$. In particular, $G_e \neq G_z$.

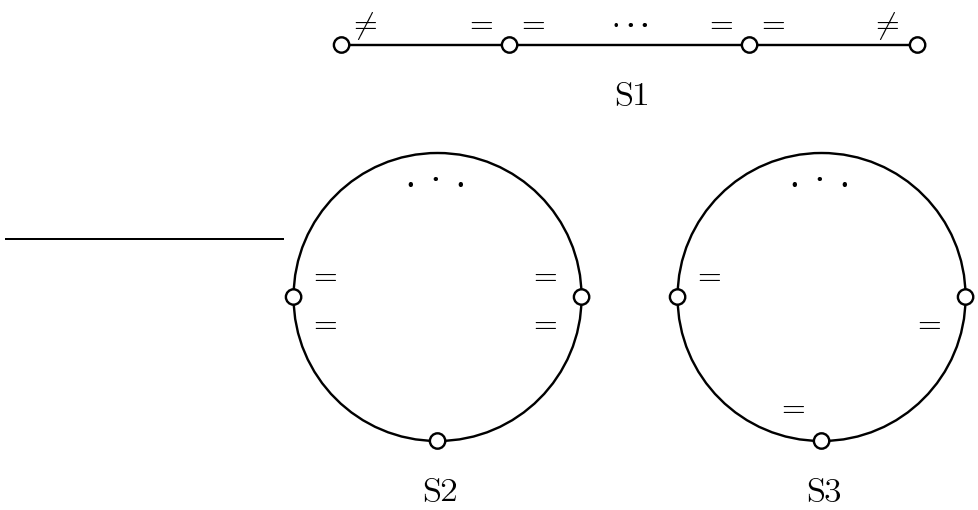

FIGURE 1. Shelters in T/G. The "=" signs denote when an inclusion $G_e \subseteq G_v$ is an equality. The "$\neq$" signs denote when an inclusion is proper. Other inclusions may or may not be proper.

For the converse, suppose that $G_e = G_v$ and $G_f = G_w$. Then clearly $G_z = G_v$, hence $G_e = G_z$. □

Now we can prove Proposition 1.13.

Proof of Proposition 1.13. It is obvious that every edge in a shelter is surviving. The converse is straight forward, but there are several cases. Some of these cases are presented in [20], but we present an entire proof for completeness. Let T by a G–tree and suppose that e is a surviving edge in T. Let T' be a reduced G–tree such that e is not collapsed in $T \to T'$. We denote the image of e in T/G by e.

case 1: The image of e in T'/G is an interval.

Let Y be the maximal subtree of T/G the contains e and is collapsed to e in $T/G \to T'/G$. We will show by induction on the number of edges in Y that Y contains a shelter of type S1 that contains e. If the number of edges in Y is one, then $Y = e$ and as T' is reduced, e is a shelter.

Now suppose that the number of edges in Y is greater than one. Let f be an edge of Y other than e. Then the image of Y in T_f/G is the maximal subtree in T_f/G that contains e and collapses to e in $T_f/G \to T'/G$. Hence, by induction the image of Y in T_f/G contains a shelter $(e_0, \dots, e_n)$ of type S1 containing e. We consider these edges as edges of T/G.

Suppose that f is adjacent to $o(e_0)$. Orient f such that $t(f) = o(e_0)$. Then by Lemma 1.14, either $(e_0, \dots, e_n)$ or $(f, e_0, \dots, e_n)$ is a shelter of type S1 for e as

$(e_0, \dots, e_n)$ is a shelter of type S1 in T_f/G. Similarly, we can find a shelter containing e of type S1 if f is adjacent to $t(e_n)$.

Next suppose that f is adjacent to $t(e_s)$ and $o(e_{s+1})$ for $0 \leq s \leq n-1$. If $t(e_s) = o(e_{s+1})$ in T/G, then by Lemma 1.14 $(e_0, \dots, e_n)$ is a shelter of type S1 for e. If not, then orient f such that $t(e_s) = o(f)$ and $o(e_{s+1}) = t(f)$. Then again by Lemma 1.14 $(e_0, \dots, e_s, f, e_{s+1}, \dots, e_n)$ is a shelter of type S1 for e.

case 2: The image of e in T'/G is a loop.

Let Y be the circuit in T/G that collapses to e in $T/G \to T'/G$. We again use induction on the number of edges in Y. Our claim in this case is that either Y is a shelter of type S2 or S3 or else Y contains part of a shelter of type S1 that contains e. If Y contains only one edge, then $Y = e$ and Y is a shelter of type S2 or S3 as T'/G is reduced. Otherwise as before, take any other edge f in Y other than e and look at the image of the circuit Y in T_f/G. This is the circuit in T_f/G that collapses to e in T'/G.

If the image of Y in T_f/G contains part of a shelter of type S1 that contains e, then proceed as in case 1 to show that Y in T/G contains a shelter of type S1 that contains e.

Otherwise, we suppose that the image of Y in T_f/G is a shelter of type S2 or S3. Suppose that $o(f) = t(e_0) = o(e_1)$ for two edges $e_0, e_1 \subseteq Y$. Then it is simple check using Lemma 1.14 that Y is shelter of type S2 or S3.

Finally, suppose that in T/G we have $t(e_0) = o(f)$ and $o(e_1) = t(f)$. Denote the image of f in T_f/G by z. First we assume that the image of Y is a shelter of type S2. If $G_{e_0} = G_z = G_{e_1}$ then by Lemma 1.14, $G_{e_0} = G_{o(f)} = G_f$ and $G_{o(e_1)} = G_{t(f)} = G_f$, hence Y is a shelter of type S2. If $G_{e_0} \neq G_z = G_{e_1}$ then either $G_{e_0} \neq G_{o(f)}$ in which case Y is a shelter of type S2 or S3 (depending on whether $G_f = G_{t(f)}$ or $G_f = G_{o(f)}$) or $G_{e_0} = G_{o(f)}$ and $G_f \neq G_{t(f)}$, in which case Y is shelter of type S2. If $G_{e_0} \neq G_z \neq G_{e_1}$ then if $G_{e_0} \neq G_{o(f)}$ and $G_{e_1} \neq G_{t(f)}$ then $Y - \{f\}$ is a shelter of type S1 that contains e. Otherwise suppose that $G_{e_1} = G_{t(f)}$, then by a similar argument as before, Y is a shelter of type S2. Now assume that the image of Y in T_f/G is a shelter of type S3. If $G_{e_0} = G_z = G_{e_1}$, then by Lemma 1.14, Y is a shelter of type S3. If $G_{e_0} = G_z \neq G_{e_1}$, then $G_{t(f)} = G_f$ and Y is a shelter of type S3.

This completes the proof of Proposition 1.13. □

As a corollary, we get a condition to check whether or not a G–tree is in W.

Corollary 1.15. *$T = T_{\mathcal{W}}$ and hence $T \in W$ if and only if T/G is a union of shelters.*

Proof. This follows immediately from Lemma 1.10 and Proposition 1.13 as $T = T_{\mathcal{W}}$ if and only if every edge in T is surviving. □

1.4. **Finite Dimensionality of W.** We conclude our treatment of deformation spaces for a general finitely generated group with a discussion of the finite dimensionality of the deformation retract W.

If T and T' are G–trees in W then there is an elementary deformation taking $T \to T'$. As each of the elementary moves is a homotopy equivalence of the quotient graph, T/G is homotopy equivalent to T'/G. Therefore homotopy invariants of graphs are invariants of deformation spaces.

Lemma 1.16. *Let $\mathcal{D}$ be a non-ascending deformation space for a finitely generated group G. Then the number of vertices in T/G for any G–tree $T \in W$ is bounded.*

Proof. By the above remark, the Euler characteristic $\chi(T/G)$ is constant for $T \in \mathcal{D}$. We can compute the Euler characteristic by $\chi(T/G) = \frac{1}{2}(V_1 - V_3 - 2V_4 - 3V_5 - \ldots)$, where V_s denotes the number of vertices with valence s. Therefore, as in [6], it suffices to show that V_1 and V_2 are bounded. Let N denote the number of conjugacy classes of maximal elliptic subgroups of G. This number is finite and depends only on $\mathcal{D}$.

By minimality of $T \in W$, every valence one vertex in T/G corresponds to a unique conjugacy class of a maximal elliptic subgroups.

By Corollary 1.15, T/G is a union of shelters. As $\mathcal{D}$ is non-ascending, only shelters of type S1 and S2 appear. Further as $\mathcal{D}$ is non-ascending if e_0, e_1 are adjacent edges in a shelter of type S2 with $t(e_0) = o(e_1) = v$ and $G_{e_0} \neq G_v$, then $G_{e_1} \neq G_v$ also.

Let v be a valence two vertex in T/G with adjacent edges e, f. Suppose that v is contained in a shelter of type S2. Then since there are no subdivision vertices in G–trees in $\mathcal{D}$, $G_e \neq G_v$ and $G_f \neq G_v$ by the above remark, hence v corresponds to a unique conjugacy class of maximal elliptic subgroups. If v is contained in a shelter of type S1, then as there are no subdivision vertices in G–trees in W, v must be the endpoint of two shelters of type S1. Hence again, $G_e \neq G_v$ and $G_f \neq G_v$ and v corresponds to a unique conjugacy class of maximal elliptic subgroups

This shows that $V_1 + V_2 \leq N$. Therefore, the number of vertices in T/G for any G–tree $T \in W$ is bounded. □

We will see that this above lemma implies that the deformation retract $W \subset \mathcal{D}$ for a non-ascending deformation space is finite dimensional. Before doing so, we look at a different setting where we can bound the number of vertices in T/G. As remarked above, the first Betti number $b_1(T/G)$ of a quotient graph of $T \in \mathcal{D}$ defines an invariant of the deformation space $\mathcal{D}$, which abusing notation, we denote by $b_1(\mathcal{D})$. If $b_1(\mathcal{D}) = 0$ then $\mathcal{D}$ is non-ascending as there are no loops in T/G for any $T \in \mathcal{D}$.

A deformation space is *locally finite* if every G–tree $T \in \mathcal{D}$ is locally finite. As modifying a locally finite G–tree by an elementary deformation results in a locally finite tree, a deformation space is locally finite if a single G–tree $T \in \mathcal{D}$ is locally finite. We remark that if $\mathcal{D}$ is locally finite (as a deformation space) then $\mathcal{D}$ and the deformation retract W are locally finite as simplicial complexes although the converse is not true.

Bass and Kulkarni introduced an invariant of a locally finite deformation space $\mathcal{D}$, called the *modular homomorphism* $q_{\mathcal{D}}\colon G \to \mathbb{Q}^\times$ [3]. This homomorphism is defined

by:

$$q_{\mathcal{D}}(g) = [V : V \cap V^g]/[V^g : V \cap V^g]$$

where V is any subgroup of G commensurable to a vertex stabilizer for a G–tree $T \in \mathcal{D}$. There is a useful alternative description of this homomorphism. For an edge $e \in E(T)$, define $i(e) = [G_{o(e)} : G_e]$ and $q(e) = i(e)/i(\bar{e}) \in \mathbb{Q}^\times$. This map q descends to edges in T/G and hence also to $H_1(T/G)$ by multiplication. The homomorphism $q_{\mathcal{D}}\colon G \to \mathbb{Q}^\times$ is the composition $G \to H_1(T/G) \to \mathbb{Q}^\times$ [18].

If $q_{\mathcal{D}}(G) \cap \mathbb{Z} = \{1\}$ then $\mathcal{D}$ is non-ascending as the modulus of any ascending loop is a non-trivial integer. The converse is not true. Forester showed that if $q_{\mathcal{D}}(G) \cap \mathbb{Z} = \{1\}$ then the canonical deformation space for a generalized Baumslag–Solitar group $\mathcal{D}$ is finite dimensional [18]. In fact Forester showed that the quotient $W/\operatorname{Out}(G)_{\mathcal{D}}$ is compact in this case. We remark that the $\mathbb{Z}$–rank of the subgroup $q_{\mathcal{D}}(G) \subseteq \mathbb{Q}^\times$ is bounded by $b_1(\mathcal{D})$.

Lemma 1.17. *Let $\mathcal{D}$ be a locally finite irreducible deformation space for a finitely generated group with $b_1(\mathcal{D}) = 1$. Then the number of vertices in T/G for any G–tree $T \in W$ is bounded.*

Proof. The proof is similar to the proof of Lemma 1.16, we must show that there is a bound on the number of valence one and two vertices. Again, let N be the number of conjugacy classes of maximal elliptic subgroups. As before, each valence one vertex corresponds to a unique conjugacy class of maximal elliptic subgroups.

If all of the shelters are type S1, then every valence two vertex must be the endpoint of the two shelters it is in and hence corresponds to a unique conjugacy class of maximal elliptic subgroups. Therefore, suppose that we have a shelter of type S2 or S3. As $b_1(\mathcal{D}) = 1$, there is only one such shelter. Any valence two vertex not in this shelter must by in a shelter of type S1 and as above it corresponds to a unique conjugacy class of maximal elliptic subgroups. Therefore, we only need to bound the number of valence two vertices in the shelter.

If the shelter is type S2, then there can be at most one valence two vertex as there are no subdivision vertices in G–trees in W. Otherwise, if the shelter is type S3, then $q_{\mathcal{D}}(G)$ is generated by an integer q and the number of valence two vertices is bounded by the number of prime factors in q as this bounds the length of a chain of proper subgroup inclusions that can appear in T/G. □

Theorem 1.18. *Let G be a finitely generated group and $\mathcal{D}$ an irreducible deformation space for G. If $\mathcal{D}$ is either:*

1. *non-ascending; or*
2. *locally finite and has $b_1(\mathcal{D}) \leq 1$;*

then the deformation retract $W \subset \mathcal{D}$ is finite dimensional.

Proof. Recall that a simplex in W is a sequence of collapse moves between G–trees in W. As a collapse move reduces the number of vertices in the quotient graph by at

least one, finite dimensionality of W is equivalent to a uniform bound on the number of vertices in T/G for any $T \in W$. This is the content of Lemmas 1.16 and 1.17. □

The first part of the previous theorem also appears as Theorem 7.6 in [20]. An example of a deformation space $\mathcal{D}$ for which W is not finite dimensional is presented in the next section (Example 2.2).

2. Deformation spaces for GBS groups

A group G that acts on a tree such that the stabilizer of any point is infinite cyclic is called a *generalized Baumslag–Solitar (GBS) group*. These groups have also recently appeared in [18, 24].

This action of a GBS group determines a graph of groups decomposition of G where all of the edge groups and vertex groups are isomorphic to $\mathbb{Z}$. As such, all of the attaching maps are given by multiplication by some nonzero integer. This data can be represented succinctly as a *labeled graph*. Specifically, a labeled graph is a pair (Γ, λ) where Γ is a finite graph and $\lambda\colon E(\Gamma) \to \mathbb{Z} - \{0\}$ is a function. The labels $\lambda(e)$ represent, for a chosen set of generators for the vertex and edge groups, the inclusion maps $G_e \to G_v$. See Figure 2 for examples of labeled graphs.

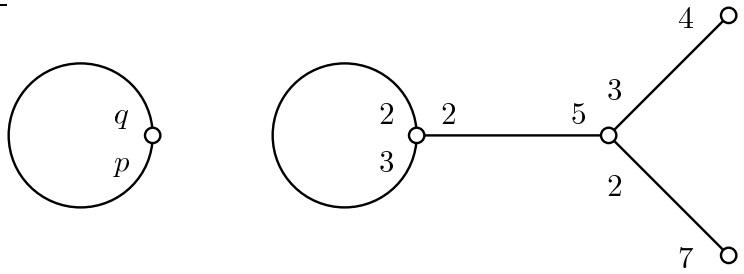

Figure 2. Examples of labeled graphs. The labeled graph on the left represents the classical Baumslag–Solitar group $BS(p, q)$.

There is a certain bit of ambiguity in the function λ resulting from different choices of generators for G_e and G_v. The different choices result in changing the signs of $\lambda(e)$ for all $e \in E_o(v)$ at some vertex v or changing the signs of $\lambda(e)$ and $\lambda(\bar{e})$ for some edge e. Such changes are called *admissible sign changes*. We consider two labeled graphs the same if they differ by admissible signs changes. Our labeled graphs are always equipped with a marking, i.e., there is G–tree T with $T/G = (\Gamma, \lambda)$ as graphs of groups. We record in Figure 3 and Figure 4 the effect of the elementary moves and the special moves listed in Section 1.1 on labeled graphs [11, 18].

A GBS group G is *non-elementary* if it is not isomorphic to $\mathbb{Z}, \mathbb{Z}^2$ or the Klein bottle group. For a non-elementary GBS group, the elliptic subgroups arising from a labeled graph are determined algebraically and do not depend on the particular tree [17]. This implies that any two (marked) labeled graphs for a GBS group are

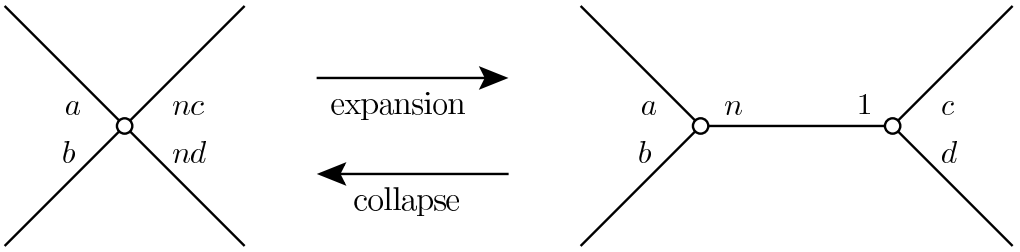

FIGURE 3. The effect of elementary moves on labeled graphs.

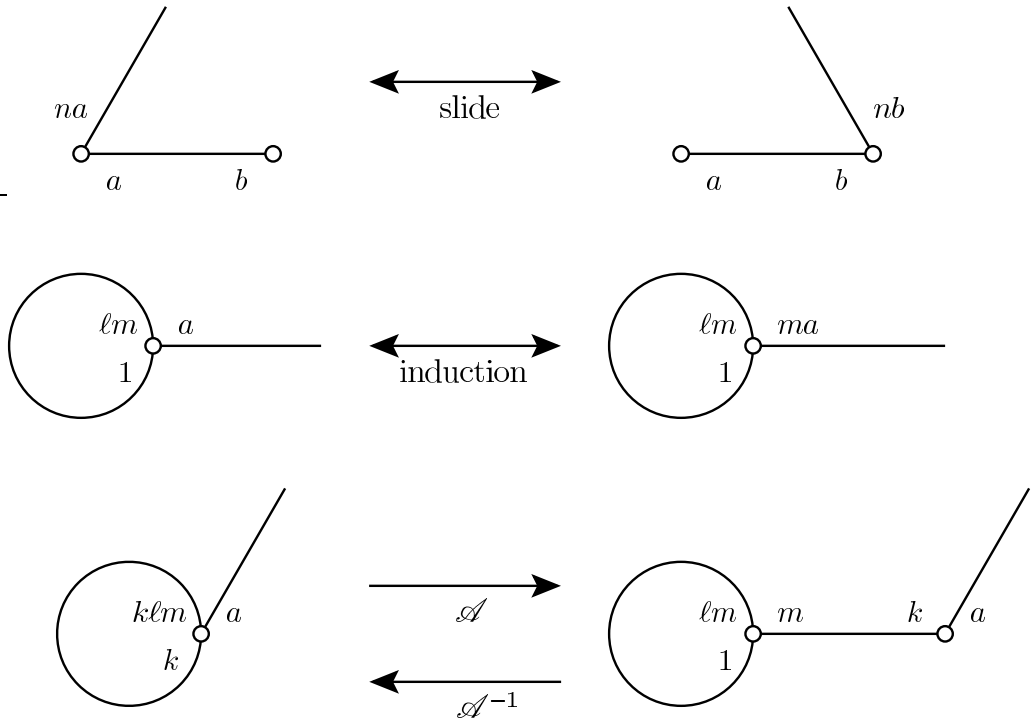

FIGURE 4. The effect of the special elementary deformations described in Section 1.1 on labeled graphs. These three moves suffice to relate any two reduced labeled graphs representing the same GBS group.

related by a sequence of elementary moves. Further, the set of conjugacy classes of elliptic subgroups is fixed by any outer automorphism. Thus if $\mathcal{D}$ is the deformation space containing one of these trees, then $\mathrm{Out}(G)_{\mathcal{D}} = \mathrm{Out}(G)$. When we speak of a deformation space for a GBS group, we will always mean this particular canonical deformation space. For another class of groups that have $\mathrm{Out}(G)_{\mathcal{D}} = \mathrm{Out}(G)$ for a particular deformation space $\mathcal{D}$ see [10].

In the following example we build a star in K and in W and describe the deformation retract $K \to W$ on these stars for the group $G = BS(2,4)$. Given a set S of vertices in a simplicial complex, the subcomplex *spanned by* S is the subcomplex consisting of all simplices whose vertices belong to the set S.

Example 2.1. Let $G = BS(2,4)$ and $\mathcal{D}$ be the canonical deformation space for $BS(2,4)$. Fix $T \in W$ that has associated labeled graph as pictured on the left in Figure 2. Pick a vertex $v \in V(T)$ and choose a generator g of G_v. Label the edges emanating from v as $E_o(v) = \{e_0, e_1, f_0, f_1, f_2, f_3\}$ where $ge_s = e_{s+1 \mod 2}$ and $gf_s = f_{s+1 \mod 4}$.

Two expansions of T are defined by using pairs $J_0 = (\langle g^2 \rangle, \{e_0, f_0, f_2\})$ and $J_1 = (\langle g^2 \rangle, \{e_0, f_1, f_3\})$. Expansion by either of these pairs results in the labeled graph in the bottom left of Figure 5. However, the marking is different. See Figure 6 for a local picture of the expansion at v to see the difference. A third expansion of T is defined by the pair $I = (\langle g^2 \rangle, \{f_0, f_2\})$. A final expansion is given by the pair $I' = (\langle g \rangle, \{f_0, f_1, f_3, f_4\})$. This is the same as the expansion using $(\langle g \rangle, \{e_0, e_1\})$. Each of the trees T^{J_0}, T^{J_1} and $T^{I'}$ can be further expanded using I. These resulting labeled graphs are pictured in Figure 5.

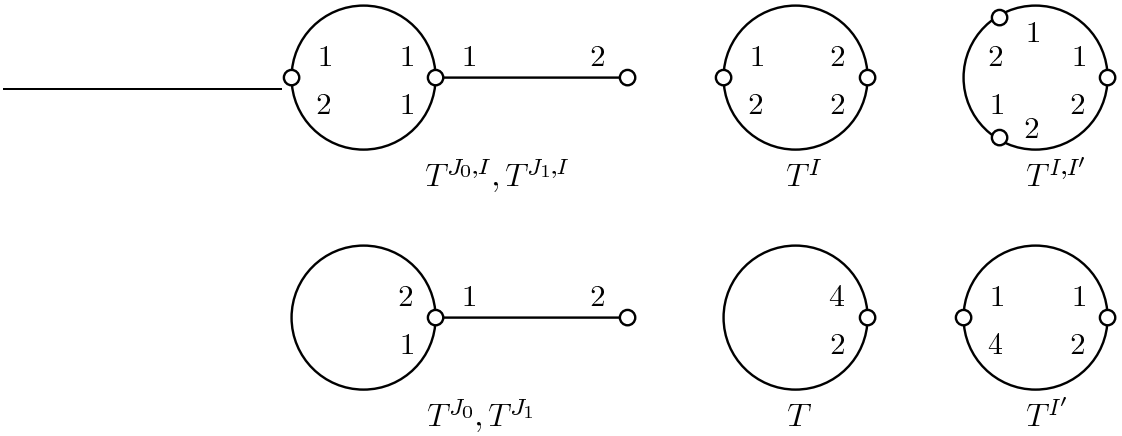

FIGURE 5. The labeled graphs for $BS(2,4)$ representing the vertices in the star of T in K.

By examining these labeled graphs is it apparent that there are no other expansions that do not create a subdivision vertex. Therefore the star of T in K is the complex pictured in Figure 7.

Of these labeled graphs, only those for T, $T^{J_0,I}$ and $T^{J_1,I}$ are covered by shelters. Therefore by Corollary 1.15, the star of T in the complex W is the two edge "V" subcomplex pictured Figure 7 spanned by the vertices labeled by $T^{J_0,I}, T, T^{J_1,I}$. The deformation retract h sends the vertices $T^{J_0}, T^I, T^{J_1}, T^{I'}$ and $T^{I,I'}$ to the vertex T, fixing the other vertices.

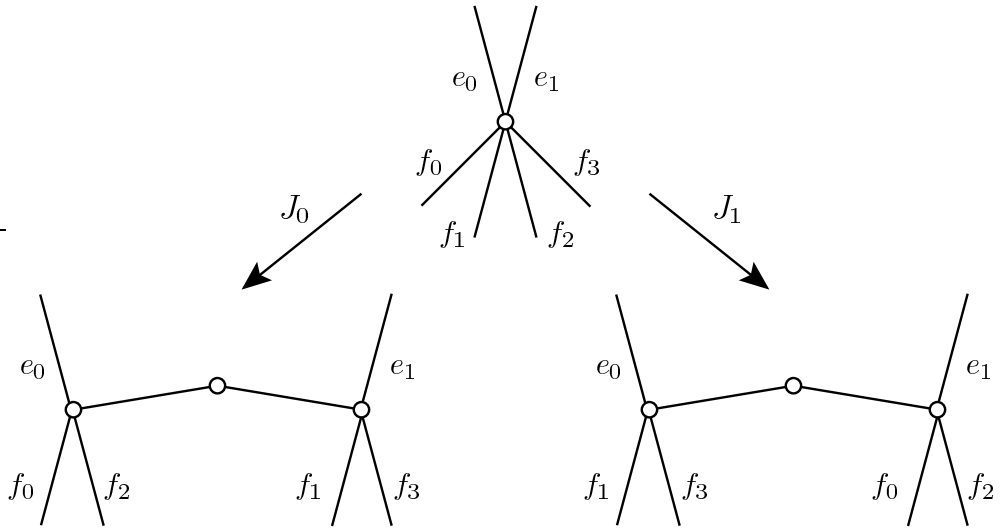

FIGURE 6. The expansions defined by J_0 and J_1.

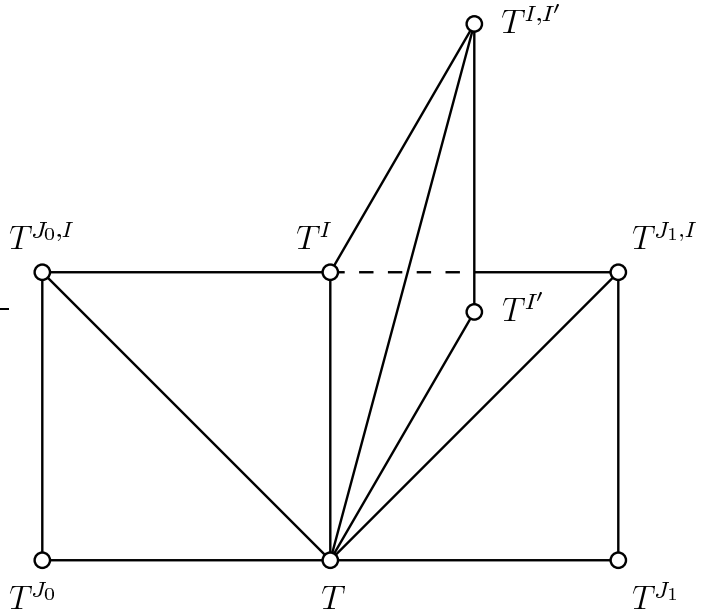

FIGURE 7. The star of T in K. The star of T in W is the "V" subcomplex spanned by $T^{J_0,I}, T, T^{J_1,I}$.

By Theorem 1.18(2) the deformation retract W in Example 2.1 is finite dimensional. We remark that both $\mathcal{D}$ and the spine K in this case are not finite dimensional. We now present an example of a deformation space $\mathcal{D}$ for which the deformation retract W is not finite dimensional.

Example 2.2. Let G be the GBS group defined by $\langle x, t_0, t_1 \mid t_0 x t_0^{-1} = t_1 x t_1^{-1} = x^2 \rangle$. This presentation is represented by the labeled graph on the left in Figure 8. Clearly the canonical deformation space $\mathcal{D}$ containing this G–tree is neither non-ascending nor has $b_1(\mathcal{D}) \leq 1$. Let T_1 be a G–tree in W with associated labeled graph pictured in the left in Figure 8. Pick a vertex $v \in V(T)$ and choose a generator g of G_v. Label the edges emanating from v are labeled $E_o(v) = \{e, e_0, e_1, f, f_0, f_1\}$ where $ge = e, gf = e, ge_s = e_{s+1 \mod 2}, gf_s = f_{s+1 \mod 2}$. Inductively, let T_k be the result of of sliding e_0 across f in T_{k-1}. The labeled graph for T_k/G is the labeled graph in the center in Figure 8.

There is a collapse sequence $T_k^k \to \cdots \to T_k^\ell \to \cdots \to T_k^0 = T_k$ where the G–trees T_k^ℓ have associated labeled graphs as in the right in Figure 8 for $0 \leq \ell \leq k$. Each of these G–trees is covered by shelters of type S3, and hence they represent vertices in W that span a k–simplex. Therefore W contains simplices of arbitrarily high dimension and is therefore not finite dimensional.

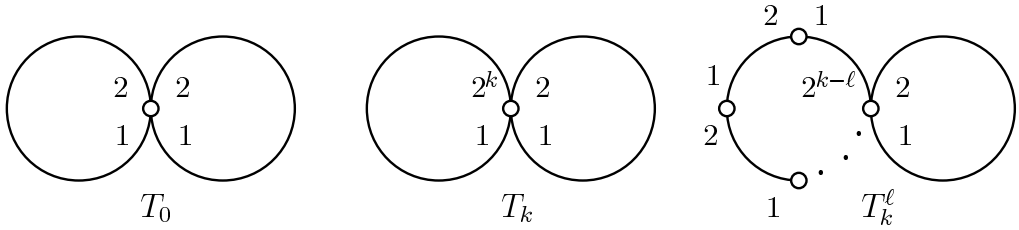

FIGURE 8. Labeled graphs representing the GBS group in Example 2.2. The deformation retract W for the canonical deformation space for this group is not finite dimensional.

In the next section we will construct the deformation retract W of the canonical deformation space for $BS(p, q)$. For another example see [9], where the deformation retract W of the canonical deformation space for the GBS group $G = \langle x, y, z \mid x^n = y^n = z^n \rangle$ for any $n \geq 2$ is constructed.

3. The canonical deformation space for $BS(p, q)$

For the remainder we use $G_{p,q}$ to denote the non-solvable Baumslag–Solitar group $BS(p, q)$ where $|q| \geq p > 1$ (see (1) in the Introduction). Our aim is to compute the automorphism groups $\mathrm{Aut}(G_{p,q})$ using the action on the canonical deformation space associated to $G_{p,q}$. We will denote this deformation space by $\mathcal{D}_{p,q}$. The deformation retract for $\mathcal{D}_{p,q}$ described in the Section 1.2 is denoted $W_{p,q}$. To begin we separate the non-solvable Baumslag–Solitar groups into three types:

1. p does not divide q;
2. $p = |q|$;
3. $q = pn$ where $|n| > 1$.

The first type was originally studied by Collins [12] in the special case that p and q do not share any factors. This case was analyzed fully by Gilbert, Howie, Metaftsis and Raptis [19]. Gilbert et al. show that if p does not divide q then the group $\mathrm{Aut}(G_{p,q})$ acts on the same $G_{p,q}$–tree that appears in Figure 2 [19]. This observation was independently discovered by Pettet [27]. Gilbert et al. analyze this action to show that $\mathrm{Out}(G_{p,q})$ is isomorphic to the dihedral group of order $2|q-p|$. A presentation for the automorphism group follows from this.

In the language of deformation spaces, the theorem of Gilbert et al. and independently Pettet translates to showing that the deformation space $\mathcal{D}_{p,q}$ is rigid when p does not divide q. A deformation space $\mathcal{D}$ is *rigid* if there is a unique reduced G–tree in $\mathcal{D}$, up to equivariant homeomorphism. Therefore, as $\mathrm{Out}(G)_{\mathcal{D}}$ acts on the deformation space, preserving the set of reduced G–trees, this unique reduced G–tree is fixed by $\mathrm{Out}(G)_{\mathcal{D}}$. Therefore $\mathrm{Out}(G)_{\mathcal{D}}$ acts on the unique reduced G–tree, extending the action of G [2]. In the case of a rigid deformation space, the complex W is a single point representing the unique reduced G–tree. The computation of Gilbert et al. is translated as the computation of the stabilizer of this point. Although we do not need it in what follows, we remark that Levitt has given a complete classification rigid deformation spaces [23] (see also [11]).

Gilbert et al. computed the automorphism group of the second type with a similar computation as for the first type [19]. Once again, the deformation space $W_{p,q}$ is rigid [23]. Thus the stabilizer of this unique reduced $G_{p,q}$–tree, is the entire group $\mathrm{Out}(G_{p,q})$. The computation of this stabilizer is similar to the computation for the other stabilizers of other $G_{p,q}$–trees that we show later on. If $p=q$ then $\mathrm{Out}(G_{p,p}) = \mathbb{Z}\rtimes(\mathbb{Z}_2\times\mathbb{Z}_2)$, if $p=-q$ then $\mathrm{Out}(G_{p,-p}) = \mathbb{Z}_{2p}\rtimes\mathbb{Z}_2$. The appearance of the $\mathbb{Z}$ factor in $\mathrm{Out}(G_{p,p})$ is due to the fact that $G_{p,p}$ has a nontrivial center. Computing the full automorphism groups from here is trivial.

The third type was studied by Collins and Levin [13] and a presentation for the group $\mathrm{Aut}(G_{p,q})$ was given by algebraic means. We will approach this using deformation spaces. By passing to an invariant tree $X_{p,q}\subseteq W_{p,q}$ ($X_{p,q}=W_{p,q}$ if $|q|/p$ is prime) we will compute $\mathrm{Out}(G_{p,q})$ via Bass–Serre theory. The presentations of $\mathrm{Out}(G_{p,q})$ and $\mathrm{Aut}(G_{p,q})$ in this case are given in Theorem 4.4. For the remainder we suppose that $q=pn$ where $p,|n|>1$.

3.1. **$G_{p,q}$–trees in $W_{p,q}$.** Using the results of Section 1.3 we will give a classification of the $G_{p,q}$–trees representing vertices in $W_{p,q}$. Denote by $\Pi(n)\subset\mathbb{Z}$ the multiplicative monoid generated by the factors of n, and $\Pi^+(n)=\{m\in\Pi(n)\,|\,m\geq 1\}$.

Lemma 3.1. *If $T\in W_{p,q}$ is reduced, then after some admissible sign changes the associated labeled graph for $T/G_{p,q}$ is either:*

1. *a single edge e with $o(e)=t(e)$ and labels $\lambda(e)=p,\lambda(\bar{e})=q$; or*
2. *two edges e,f with $o(e)=t(e)=o(f)$ and labels $\lambda(e)=1,\lambda(\bar{e})=n,\lambda(f)=m\neq 1$ where $m\in\Pi^+(n)$ and $\lambda(\bar{f})=p$.*

See Figure 9.

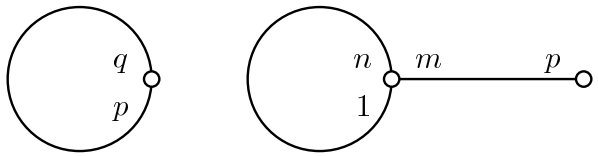

FIGURE 9. Labeled graphs representing reduced $G_{p,q}$–trees in $W_{p,q}$, $m \in \Pi^+(n), m \neq 1$.

Proof. We will show that the collection of such labeled graphs is closed under slides, inductions and $\mathscr{A}^{\pm 1}$–moves. By Theorem 1.6, this implies the conclusion of the lemma.

If T is as in case 1, the only possible move is an $\mathscr{A}$–move. This results in a labeled graph as described in case 2, except possibly $\lambda(f)$ could be negative. If this factor is negative, changing the signs of the labels on the edge f and then changing the sign of $\lambda(\bar{f})$ results in a labeled graph as in case 2.

If T is as in case 2 then a slide move possibly followed by admissible sign changes as above either results in a labeled graph as in case 2 with $\lambda(f) = |nm|$ or $|m/n|$. The latter case is only possible when n properly divides $\lambda(f)$ as otherwise the resulting labeled graph is not reduced. An induction move only changes the label $\lambda(f)$, which is either multiplied or divided by a factor of n. The latter case is only possible when this factor is not equal to $\lambda(f)$ as otherwise the resulting labeled graph is not reduced. Thus again after possible admissible sign changes the resulting labeled graph is also as in case 2. An $\mathscr{A}^{-1}$–move is only possible when $\lambda(f) \leq |n|$. The resulting labeled graph is as in case 1. An $\mathscr{A}$–move is not possible in this case. □

Given such labeled graphs as in case 2 of Lemma 3.1, we say an induction move is *increasing* if the label $|\lambda(f)|$ is larger after the induction move and *decreasing* otherwise. Similarly define *increasing slides* and *decreasing slides*.

Lemma 3.2. *If $T \in W_{p,q}$, then after some admissible sign changes the associated labeled graph for $T/G_{p,q}$ is either as in case 1 of Lemma 3.1 or it consists of a shelter γ of type S3 with a single edge f attached at $o(f)$. The labels on f are $\lambda(f) = m$ where $m \in \Pi^+(n)$ and $\lambda(\bar{f}) = p$.*

Proof. First, suppose the labeled graph representing T is a cycle γ. If γ is covered by shelters of type S1, then γ collapses to a cycle with at least two edges, contradicting Lemma 3.1. If γ is a shelter of type S3 then after collapsing every edge except one, we get a reduced labeled graph that is an ascending loop, contradicting Lemma 3.1. If γ is a shelter of type S2, then as there are no subdivision vertices in $G_{p,q}$–trees in $W_{p,q}$, hence γ is a single edge and hence must be as in case 1 of Lemma 3.1.

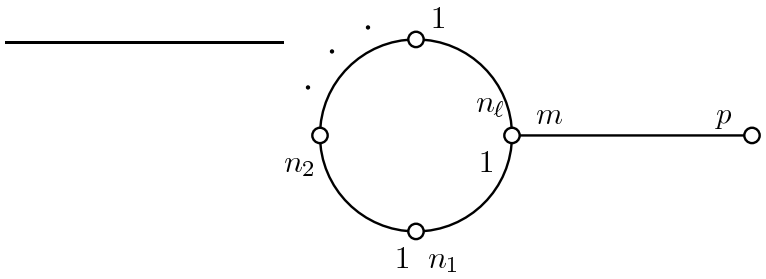

FIGURE 10. Labeled graphs representing $G_{p,q}$–trees in $W_{p,q}$. The labels on the circuit γ satisfy $n_1 \cdots n_\ell = n$ and $|n_s| \neq 1$ with the possible exception of n_ℓ.

Otherwise, the labeled graph consists of a circuit γ with some finite trees attached. Since there is a unique conjugacy class of maximal elliptic subgroups for $G_{p,q}$, there is at most one valence one vertex in a labeled graph representing a $G_{p,q}$–tree in $W_{p,q}$. Therefore, there is at most one finite tree, F, attached to γ and it is linear. As there is a unique maximal conjugacy class of elliptic subgroups and as $G_{p,q}$–trees in $W_{p,q}$ do not subdivision vertices, F cannot have any valence two vertices and is hence a single edge f.

First we suppose γ contains a shelter of type S1. Then γ is covered by disjoint shelters of type S1, further there are at least two shelters of type S1 needed to cover γ. Therefore, we can reduce the labeled graph for T to a get a reduced labeled graph with a circuit with at least two edges. By Lemma 3.1, this is a contradiction. Therefore, γ is either a type S2 or type S3 shelter.

Now suppose γ is a type S2 shelter. Since there are no subdivision vertices and there is a single edge attached, γ is either one or two edges. If γ is a single edge, then as f must be in a shelter and using Lemma 3.1 we see that the labels on γ are 1 and n. Hence γ can also be thought of as a shelter of type S3. If γ is two edges then a similar argument shows γ can be thought of as a type S3 shelter. Therefore we have shown that the labeled graph is shelter of type S3 with a single edge attached.

To see which labels can appear on f we collapse the cycle γ to a single edge. The labels appearing on γ are all factors of n, therefore collapsing γ cannot change whether or not $\lambda(f) \in \Pi(n)$. Since by Lemma 3.1, $\lambda(f) \in \Pi^+(n)$ after collapsing γ, we must have that $f \in \Pi^+(n)$ initially. Also collapsing γ does not change $\lambda(\bar{f})$. Again, by Lemma 3.1 $\lambda(\bar{f}) = p$ initially. □

3.2. **The complex $X_{p,q}$.** To see the motivation behind the definition of $X_{p,q}$ we will first describe the complex $W_{p,q}$ when $|n|$ is prime. As $|n|$ is prime, by Lemma 3.2, any $G_{p,q}$–tree in $W_{p,q}$ has associated labeled graph as pictured in Figure 11. We continue to use the notation from Lemma 3.1 to denote the edges in the reduced labeled graphs.

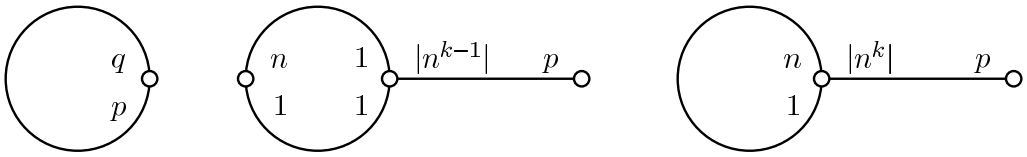

FIGURE 11. Labeled graphs representing $G_{p,q}$–trees in $W_{p,q}$ when $|n|$ is prime ($k \geq 1$).

As the longest collapse sequence between the labeled graphs in Figure 11 is one, the complex $W_{p,q}$ is one dimensional, hence a tree. To each of the reduced labeled graphs in Figure 11 we assign a non-negative integer called the *level*. For the one edge reduced labeled graph, the level is 0. For the two edge reduced labeled graphs, the level is the non-negative integer k appearing in the exponent of n. We will also use the term level when talking about a reduced $G_{p,q}$–tree T as the level of the associated labeled graph $T/G_{p,q}$.

As in Example 2.1, the star of a reduced $G_{p,q}$–tree with level 0 is a p–pod. This is not the case when $|n|$ is composite, although we will see in Lemma 3.8 that in this case the star is very close to being a p–pod. The terminal vertices of the p–pods are represented by labeled graphs pictured in the center of Figure 11 with $k = 1$. In addition to collapsing to a $G_{p,q}$–tree with level 0, these trees also collapse to a $G_{p,q}$–tree with level 1.

The star of a reduced $G_{p,q}$–tree with level $k \geq 1$ is a $(|n|+1)$–pod. This follows as there are $|n|$ decreasing slides of the edge f counterclockwise around the loop $\bar{e}$ to a reduced tree with level $k-1$ (further collapsing resulting in an $\mathscr{A}^{-1}$–move is needed if $k = 1$) and a unique increasing slide of the edge f clockwise around the loop e to a reduced tree with level $k+1$. Figure 12 shows a piece of the complex $W_{2,4}$. The entire complex $W_{2,4}$ is comprised of similar pieces, glued 2 at a time along a level 0 vertex, such that the resulting complex is contractible.

We define $X_{p,q}$ to mimic $W_{p,q}$ in the case that $|n|$ is prime.

Definition 3.3. Suppose $q = pn$ where $p, |n| > 1$. Let $X_{p,q}$ be the subcomplex of $W_{p,q}$ spanned by $T \in W_{p,q}$ whose associated labeled graph is one of the three pictured in Figure 11, $k \geq 1$.

It is clear that $X_{p,q} = W_{p,q}$ when $|q|/p$ is prime. In any case, the longest collapse sequence between such labeled graphs is one. Hence the subcomplex $X_{p,q}$ is a (possibly disconnected) graph. The notion of level defined for reduced labeled graphs in $W_{p,q}$ when $|n|$ is prime extends to reduced labeled graphs in $X_{p,q}$. We will show that $X_{p,q}$ is a tree. First we show that $X_{p,q}$ is connected.

Lemma 3.4. *If a, b divide n and the product ab also divides n then the composition of the increasing inductions using the subgroups of index a and b is the increasing*

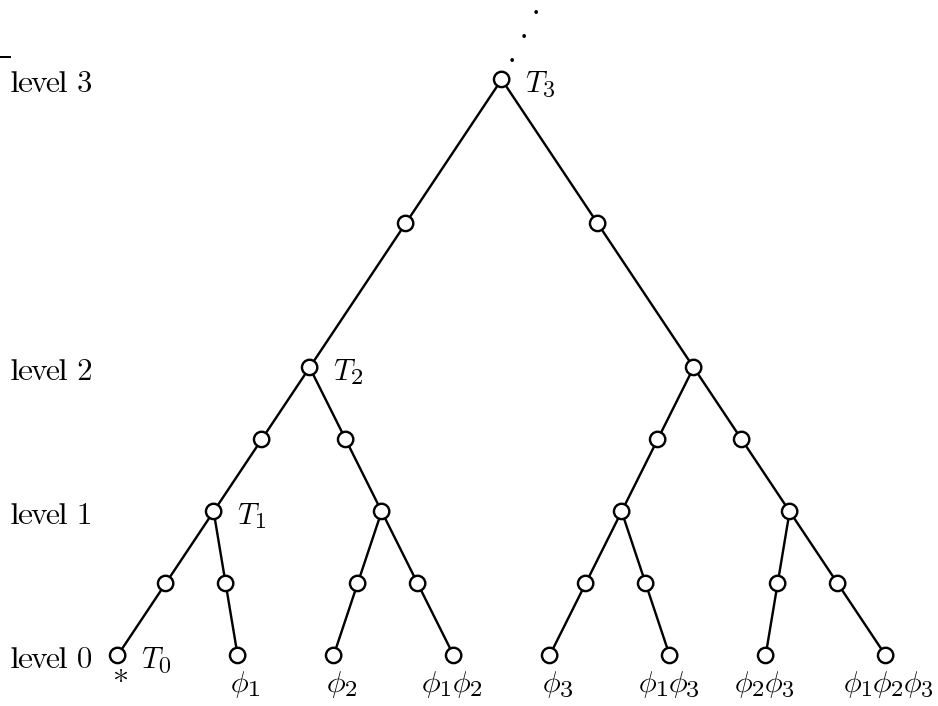

FIGURE 12. A piece of the complex $W_{2,4}$. The labels on the bottom show the action of the automorphisms ϕ_k listed in (4).

induction using the subgroup of index ab. Further, if $ab = n$, then the composition of the increasing inductions is an increasing slide.

Proof. Write $n = ab\ell$. The expansion appearing in an increasing induction depends only on the subgroup of the vertex group. As there is a unique subgroup of any given index in $\mathbb{Z}$, the expansion is uniquely defined by the expansion in the labeled graph. The 2–cell in $W_{p,q}$ pictured in Figure 13 shows that the result of the composition of increasing inductions using the individual subgroups of index a and b is the same as the result of the increasing induction using the the subgroup of index ab. If $\ell = 1$, then the sequence of moves displayed along the bottom is increasing slide. □

Lemma 3.5. *Let $T \in W_{p,q}$ be reduced and suppose $T' \in W_{p,q}$ is obtained by the composition of the increasing inductions using the subgroups of index a and then b and $T" \in W_{p,q}$ is obtained by the composition of the increasing inductions using the subgroups of index b and then a. Then $T' = T''$.*

Proof. If the product ab divides n, then either composition of the increasing inductions is the increasing induction using the subgroup of index ab by Lemma 3.4. Hence the Lemma follows in this case.

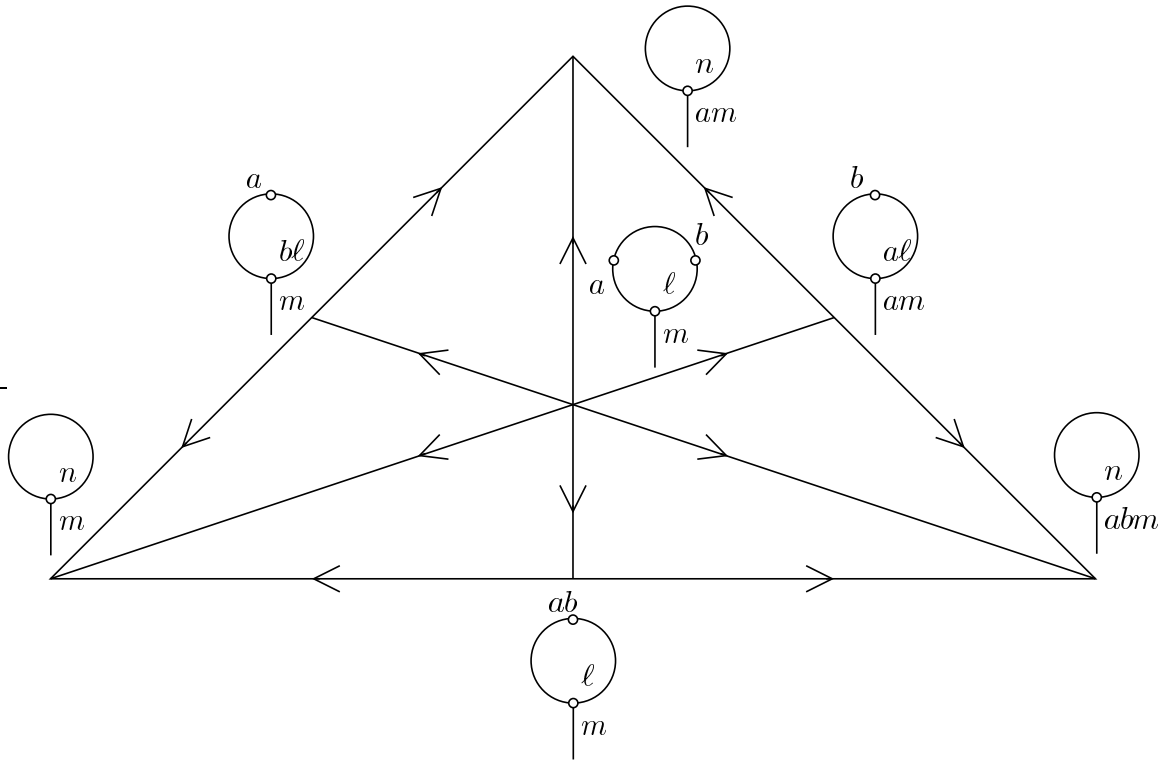

FIGURE 13. The 2–cell in $W_{p,q}$ in Lemma 3.4. The labels 1 and p are omitted from the figure.

For the other case write $a = a'\ell$, $b = b'\ell$ where $\ell = \gcd(a, b)$. Then $a'b'$ divides n. The composition of the increasing inductions using a and then b can be written as the composition of the four increasing inductions using ℓ, a', b' and then ℓ. Applying the earlier observation to the pair a', b' we can write the composition of the increasing inductions as the the composition of the four increasing inductions using ℓ, b', a' and then ℓ. But since these combine to give us the composition of the increasing inductions using $b'\ell = b$ and $a'\ell = a$ the Lemma follows. □

Define a map on reduced $G_{p,q}$–trees in $W_{p,q}$ to reduced $G_{p,q}$–trees in $X_{p,q}$ by letting $x(T)$ be the tree obtained by a composition of increasing inductions resulting in changing $\lambda(f) = m$ to $mm' = n^k$ where n does not divide m'. By Lemma 3.5 x is well-defined. Denote the integer m' by $\lambda(T)$.

Lemma 3.6. *If $T, T' \in W_{p,q}$ are related by a slide or an induction then $x(T) = x(T')$ or $x(T)$ and $x(T')$ are related by a slide.*

Proof. As by Lemma 3.4 an increasing slide is a special case of an increasing induction we only need to show the lemma when T' is obtained from T by an increasing induction.

If ℓ divides $\lambda(T)$, then $\lambda(T') = \lambda(T)/\ell$ and by Lemma 3.5 $x(T) = x(T')$ and the lemma holds.

If ℓ does not divide $\lambda(T)$, then $\lambda(T') = n\lambda(T)/\ell$. By Lemma 3.5 the composition of the increasing inductions $T \to T' \to x(T')$ using the factors ℓ and then $\lambda(T')$ can be written as the composition of the increasing inductions $T \to x(T) \to x(T')$ using the factors $\lambda(T)$ and then n. Thus by Lemma 3.4, $x(T)$ and $x(T')$ are related by a slide. □

Proposition 3.7. *The graph $X_{p,q}$ is connected.*

Proof. Let T and T' be reduced $G_{p,q}$–trees in $X_{p,q}$. Then there is a 1–skeleton path in $W_{p,q}$ that connects these two trees. Moreover, by Theorem 1.6 we can assume that this path is given by a sequence of slides, inductions and $\mathscr{A}^{\pm 1}$–moves between reduced trees.

By inserting an increasing induction move before any $\mathscr{A}^{-1}$–move and after any $\mathscr{A}$–move, we can assume that any $\mathscr{A}^{\pm 1}$–move appearing is between reduced trees in $X_{p,q}$. Therefore we only need to show that any two reduced G–trees in $X_{p,q}$ related by a sequence of slide and inductions between reduced $G_{p,q}$–trees in $W_{p,q}$ are in related by a sequence of slides between reduced $G_{p,q}$–trees in $X_{p,q}$. By Lemma 3.6 the map x transforms such a sequence of inductions and slides into a sequence of slides between reduced $G_{p,q}$–trees in $X_{p,q}$. □

Therefore $X_{p,q}$ is a connected graph. The $G_{p,q}$–trees in $X_{p,q}$ form pieces in $X_{p,q}$ similar to the one pictured in Figure 12. This is true as from any reduced $G_{p,q}$–tree with level $k \geq 1$ there are $|n|$ decreasing slides resulting in a reduced $G_{p,q}$–tree with level $k-1$ (further collapsing resulting in an $\mathscr{A}^{-1}$–move is needed if $k = 1$) and a unique increasing slide resulting in a reduced $G_{p,q}$–tree with level $k+1$. In the next lemma we look at the link of a $G_{p,q}$–tree with level 0.

Lemma 3.8. *If $T \in W_{p,q}$ is a reduced with level 0 then the link of T in $W_{p,q}$ is homotopy equivalent to a set of p points. Moreover, each one of these points is naturally identified to an adjacent $G_{p,q}$–tree in $X_{p,q}$.*

Proof. By Lemma 3.2, labeled graphs for $G_{p,q}$–trees in the link of T all consist of a shelter of type S3 attached to a single edge f at some vertex $o(f) = v$. The labels on f are $\lambda(f) = 1$ and $\lambda(\bar{f}) = p$. Starting from v the labels on the shelter are $(1, n_1, 1, n_2, \ldots, 1, n_\ell)$ where $n_1 n_2 \cdots n_\ell = n$ and $|n_s| \neq 1$ with the possible exception of n_ℓ. See Figure 10.

We will now define a deformation retraction of the link using Quillen's Poset Lemma (Lemma 1.11). This retraction is the composition of two maps. If $|n_\ell| \neq 1$, there is an elementary move that expands this $G_{p,q}$–tree T' to a $G_{p,q}$–tree $h_0(T')$ where

$n_m = 1$. If $|n_\ell| = 1$, define $h_0(T') = T'$. Then h_0 defines a poset map on the link that satisfies $h_0(T') \geq T'$. Thus Quillen's Poset Lemma applies and defines a deformation retraction of the link.

For $G_{p,q}$–trees T' in the image of h_0 define $h_1(T')$ as the $G_{p,q}$–tree found by collapsing the edges in the shelter of type S3 that are not adjacent to v. Then h_1 defines a poset map on the image of h_0 satisfying $h_1(T') \leq T'$. Therefore Quillen's Poset Lemma applies again. The image of $h_1 h_0$ are the p $G_{p,q}$–trees with labeled graph a shown in the middle of Figure 11 where $k = 1$. This proves the lemma. □

Using this lemma, we can prove that $X_{p,q}$ is a simply-connected, hence a tree.

Theorem 3.9. *The subcomplex $X_{p,q} \subseteq W_{p,q}$ is a tree.*

Proof. As $X_{p,q}$ is a connected graph, we just need to show that $X_{p,q}$ does not contain a circuit, i.e. an edge path homeomorphic to a circle. Suppose there is a circuit $\gamma \subset X_{p,q}$.

Since $W_{p,q}$ is simply-connected (Theorem 1.7), γ bounds a disk D in $W_{p,q}$. By Lemma 3.8 and since γ is homeomorphic to a circle, γ cannot contain a vertex corresponding to a $G_{p,q}$–tree of level 0. Therefore, γ crosses some $G_{p,q}$–tree $T \in X_{p,q}$ with level $k \geq 1$ which is minimal among all $G_{p,q}$–tree along γ. But as there is a unique edge from a $G_{p,q}$–tree with level k to a $G_{p,q}$–tree with level $k+1$, this implies that the two edges of γ adjacent to this tree are the same and therefore γ is not a circuit. Hence $X_{p,q}$ does not contains a circuit. □

Question 3.10. Is there a poset map that defines a deformation retraction from $W_{p,q} \to X_{p,q}$? If so, can one define this map for a general deformation space to get a further deformation retraction of $\mathcal{D}$?

4. Computation of $\mathrm{Out}(BS(p,q))$

In this section we will use the action of $\mathrm{Out}(G_{p,q})$ on the tree $X_{p,q}$ to give a presentation of this group in the case that p properly divides q. The vertices in $X_{p,q}$ corresponding to $G_{p,q}$–trees that are not reduced are subdivision points, removing these from $X_{p,q}$ does not alter the action of $\mathrm{Out}(G_{p,q})$. We begin by describing the quotient $X_{p,q}/\mathrm{Out}(G_{p,q})$.

4.1. **The quotient $X_{p,q}/\mathrm{Out}(G_{p,q})$.** The following proposition is a restatement of a special case of Proposition 5.3 in [31] (cf. [1]).

Proposition 4.1. *Suppose G is a GBS group and T, T' are G–trees with infinite cyclic point stabilizers. If the associated labeled graphs for T/G and T'/G are isomorphic, then there is an outer automorphism $\Phi \in \mathrm{Out}(G)$ such that $T\Phi = T'$.*

Therefore, the quotient $X_{p,q}/\mathrm{Out}(G_{p,q})$ can be identified with the ray $[0, \infty)$, where the integer point k is represented by an (unmarked) labeled graph with level k. See Figure 14.

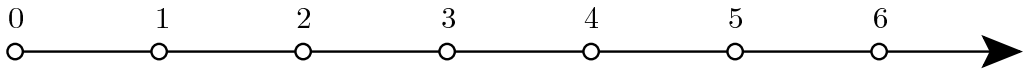

FIGURE 14. The quotient graph $X_{p,q}/\operatorname{Out}(BS(p,q))$.

4.2. **Infinite generation of** $\operatorname{Out}(BS(p,q))$**.** Before we compute the vertex stabilizers of the tree $X_{p,q}$ we give a geometric argument showing $\operatorname{Out}(G_{p,q})$ is not finitely generated when p properly divides q. This follows easily from the following lemma.

Lemma 4.2. *Let G be a finitely generated group acting by simplicial automorphisms on a connected simplicial complex X. Then, for any point $x \in X$, there is a compact set $C \subset X$ containing x such that GC is connected.*

Proof. This is Brown's finiteness criteria for type FP_0 [7]. It is easy to prove in this case. Without loss of generality, we can assume that $x \in X^{(0)}$. Let $\{g_0, \dots, g_m\}$ be a finite generating set for G. Then take C to be the union of the 1–skeleton paths between x and $g_s x$ for $s = 0, \dots, m$. Since the g_s generate G, the set GC is connected. □

Now to see that $\operatorname{Out}(G_{p,q})$ is not finitely generated, we apply the above lemma to any $T \in X_{p,q}$ with level 0. In any compact set $C \subset X_{p,q}$ containing T there is a $k \geq 0$ such that any $G_{p,q}$–tree in C has level at most k. There is another T' with level 0 such that the geodesic from T to T' to passes through a $G_{p,q}$–tree with level $k+1$. By Proposition 4.1, $T' \in \operatorname{Out}(G_{p,q})C$. However, as $X_{p,q}$ is a tree, any path from T to T' must go through a $G_{p,q}$–tree with level $k+1$. Such a path cannot lie entirely in $\operatorname{Out}(G_{p,q})C$ as the action of $\operatorname{Out}(G_{p,q})$ preserves the level of a $G_{p,q}$–tree. In terms of Bestvina–Brady discrete Morse theory [5], the descending links of the $G_{p,q}$–trees with level $k \geq 1$ are disconnected. Hence we get the following theorem, also noted by Collins and Levin [13].

Theorem 4.3. *If $p > 1$ and p properly divides q, then $\operatorname{Out}(BS(p,q))$ is not finitely generated.*

4.3. **Vertex stabilizers in** $X_{p,q}$**.** The quotient $X_{p,q}/\operatorname{Out}(G_{p,q})$ is a ray. Lift this ray to a ray in $X_{p,q}$. Denote the $G_{p,q}$–tree on this ray representing the integer point k by T_k. Without loss of generality, we can assume $T_0/G_{p,q}$ gives rise to the presentation in (1). Further, T_k for $k \geq 1$ give rise to presentations:

$$G_{p,q} = \langle a, b_k, t \,|\, a^p = b_k^{n^k}, t b_k t^{-1} = b_k^n \rangle \tag{2}$$

where $a \mapsto x, b_k \mapsto t^{-k} x^p t^k$ and $t \mapsto t$.

Let $H_k \subseteq \operatorname{Out}(G_{p,q})$ be the stabilizer of T_k. We have two cases, depending on whether $k = 0$ or $k \geq 1$. The important fact we use is that if $\phi \in \operatorname{Aut}(G)$ fixes an irreducible G–tree T (here G can be any finitely generated group), then there is a unique ϕ–equivariant simplicial automorphism $f_\phi \colon T \to T$ [1]. Thus we get an action

of the stabilizer $H_T \subseteq \mathrm{Aut}(G)$ on the tree T that extends the action of G (viewing $G/Z(G) = \mathrm{Inn}(G)$ as a subgroup of H_T).

For an irreducible G–tree T, there are special automorphisms that fix T. These are called *twists* as they generalize the familiar notion of Dehn twist when the G–tree arises from a simple closed curve on a surface. We will only look a one type of twist, the one that corresponds to a nonseparating curve, for a general discussion of twists in G–trees, see [22]. Let e be a one edge loop in T/G with vertex v and stable letter t, when viewing e as giving rise to an HNN-extension. Then for $z \in G_v$ such that $zg = gz$ for all $g \in G_e$ the map that sends $t \mapsto zt$ is a twist in G–tree T. To fix some notation, for $g \in G$ we denote the inner automorphism $g' \mapsto gg'g^{-1}$ by c_g.

case 1: $k = 0$.

We claim that H_0 is isomorphic to the dihedral group $\mathbb{Z}_{p|n-1|} \rtimes \mathbb{Z}_2$, generated by the following automorphisms:

$$\begin{array}{rlll} \psi\colon & x & \mapsto x & \\ & t & \mapsto xt & \end{array} \qquad \begin{array}{rlll} \iota\colon & x & \mapsto x^{-1} & \\ & t & \mapsto t & \end{array} \tag{3}$$

Notice that $\psi^{p(n-1)} = c_x^{-p}$ and $\iota\psi = \psi^{-1}\iota$. Using normal forms for HNN-extensions, it is easy to see that the outer automorphism class of ψ^ℓ for $1 \leq \ell < p|n-1|$ is non-trivial. Hence the image of $\langle \psi, \iota \rangle$ in $\mathrm{Out}(G_{p,q})$ is the dihedral group $\mathbb{Z}_{p|n-1|} \rtimes \mathbb{Z}_2$. The automorphism ψ is a twist as described above for T_0, hence it fixes T_0. It is clear that the automorphism ι fixes any $G_{p,q}$–tree in $\mathcal{D}_{p,q}$. Thus the image of the subgroup $\langle \psi, \iota \rangle$ is contained in H_0. The claim that this is an equality follows exactly as the computation of Gilbert et al. for the case when p does not properly divide q [19]. This computation is similar to the computation in the next case, thus we omit it. □

case 2: $k \geq 1$.

In this case, we claim that H_k is isomorphic to the dihedral group $\mathbb{Z}_{|n^k(n-1)|} \rtimes \mathbb{Z}_2$, generated by the following automorphisms:

$$\begin{array}{rlll} \phi_k\colon & x & \mapsto x & \\ & t & \mapsto (t^{-k}x^pt^k)t & \end{array} \qquad \begin{array}{rlll} \iota\colon & x & \mapsto x^{-1} & \\ & t & \mapsto t & \end{array} \tag{4}$$

Notice that $\phi_{k+1}^n = \phi_k$ for $k \geq 1$, $\phi_1^n = \psi^p$ and $\iota\phi_k = \phi_k^{-1}\iota$. To prove this claim, it is easier to use the presentations for $G_{p,q}$ in (2). With this generating set, the automorphisms in (4) are:

$$\begin{array}{rlll} \phi_k\colon & a & \mapsto a & \\ & b_k & \mapsto b_k & \\ & t & \mapsto b_kt & \end{array} \qquad \begin{array}{rlll} \iota\colon & a & \mapsto a^{-1} & \\ & b & \mapsto b_k^{-1} & \\ & t & \mapsto t & \end{array} \tag{5}$$

Viewing these presentations as HNN-extensions $\langle a, b_k \rangle *_{\langle tb_kt^{-1} = b_k^{n^k} \rangle}$ and using normal forms for HNN-extensions is easy to see that the outer automorphism class of ϕ_k^ℓ is non-trivial for $1 \leq \ell < |n^k(n-1)|$. Hence the image of $\langle \phi_k, \iota \rangle$ in $\mathrm{Out}(G_{p,q})$ is the

dihedral group $\mathbb{Z}_{|n^k(n-1)|} \rtimes \mathbb{Z}_2$. Also with these presentations, it is apparent that ϕ_k is a twist of the $G_{p,q}$–tree T_k. Thus the image of the subgroup $\langle \phi_k, \iota \rangle$ is contained in H_k. The action of the automorphisms ϕ_k for $G_{2,4}$ on $X_{2,4}$ is shown in Figure 12.

Now suppose that $\alpha \in H_k$. Then lifting α to $\mathrm{Aut}(G_{p,q})$, we have a α–equivariant simplicial automorphism $f_\alpha \colon T_k \to T_k$. There are two type of vertices in T_k: those that are lifts of $o(f) = v$ or those that are lifts of $t(f) = w$. Lifts of v belong to an axis of a hyperbolic element of length one and lifts of w do not, thus f_α sends lifts of v to lifts of v and similar for w. Therefore, after composing α with an inner automorphism, we can assume that f_α fixes some lift of v in T with stabilizer $\langle b_k \rangle$, which we continue to denote v. Further, we can assume that v is adjacent to the unique vertex stabilized by $\langle a \rangle$. This implies that the axis of t contains v.

We label the edges emanating from v by $E_o(v) = \{e, e_0, \ldots, e_{|n|-1}, f_0, \ldots, f_{|n|^k-1}\}$ where $b_k e = e, b_k e_s = e_{s+1 \mod |n|}$ and $b_k f_s = f_{s+1 \mod |n^k|}$. Assume that $t(f_0)$ is stabilized by $\langle a \rangle$ and $tv = t(e_0)$. Define $\beta = c_{b_k} \phi_k^{n-1}$. Then $\beta(a) = b_k a b_k^{-1}$ and fixes b_k and t. After composing α with $\beta^{m'}$ for some m' we can assume that f_α fixes the edge f_0. Hence, after composing with ι we can assume that $\alpha(a) = a$ and $\alpha(b_k) = b_k$.

Therefore f_α permutes the edges e_s for $s = 0, \ldots, |n| - 1$. Now $\alpha(t)v = \alpha(t) f_\alpha(v) = f_\alpha(tv) = t(e_s) = b_k^s tv$ for some s. Thus $t^{-1} b_k^{-s} \alpha(t) \in G_v = \langle b_k \rangle$. Therefore $t^{-1} b_k^{-s} \alpha(t) = b_k^m$. Rewriting, we have $\alpha(t) = b_k^s t b_k^m = b_k^{s+mn} t = \phi_k^{s+mn}(t)$. Thus $\alpha = \phi_k^{s+mn}$ and H_k is as claimed. □

Since T_{k+1} is the unique $G_{p,q}$–tree of level $k+1$ adjacent to T_k we have inclusions $H_k \subseteq H_{k+1}$ for $k > 0$. Therefore, as a graph of groups, the infinite ray $X_{p,q}/\mathrm{Out}(G_{p,q})$ collapses to a segment with one vertex corresponding to T_0 and the other vertex corresponding to the end represented by $(T_1, T_2, \ldots)$. The stabilizer of this end is the direct limit:

$$\lim_{\to} \mathbb{Z}_{|n^k(n-1)|} \rtimes \mathbb{Z}_2 = \mathbb{Z}[\frac{1}{|n|}]/|n(n-1)| \rtimes \mathbb{Z}_2.$$

In the above, $1 \in \mathbb{Z}[\frac{1}{|n|}]/|n(n-1)|$ corresponds to the outer automorphism class of ϕ_1.

4.4. **Presentations.** The computations from Sections 4.1 and 4.3 give the presentation for $\mathrm{Out}(BS(p,q))$ appearing in the following theorem. The presentation for $\mathrm{Aut}(BS(p,q))$ follows routinely from this. This presentation was also found with an algebraic computation by Collins and Levin [13].

Theorem 4.4. *Let $q = pn$ where $p, |n| > 1$. The automorphism group $\mathrm{Aut}(BS(p,q))$ is generated by the automorphisms c_x, c_t, ψ, ι, and ϕ_k for $k \geq 1$ subject to the following*

relations:

$$\begin{array}{rl} c_t c_x^p c_t^{-1} = c_x^q & \iota^{-1} = \iota \\ \iota c_x \iota = c_x^{-1} & \iota c_t \iota = c_t \\ \iota\psi\iota = \psi^{-1} & \iota\phi_k\iota = \phi_k^{-1}, \text{ for } k \geq 1 \\ \psi^p = \phi_1^n & \psi^{p(n-1)} = c_x^{-p} \\ \psi c_x \psi^{-1} = c_x & \psi c_t \psi^{-1} = c_x c_t \\ \phi_k c_x \phi_k^{-1} = c_x, \text{ for } k \geq 1 & \phi_k c_t \phi_k^{-1} = c_t^{-k} c_x^p c_t^{k+1}, \text{ for } k \geq 1 \\ \phi_{k+1}^n = \phi_k, \text{ for } k \geq 1 & \end{array}$$

The outer automorphism group has presentation:

$$\mathrm{Out}(BS(p,q)) = (\mathbb{Z}_{|p(n-1)|} *_{\mathbb{Z}_{|n-1|}} \mathbb{Z}[\frac{1}{|n|}]/|n(n-1)|\mathbb{Z}) \rtimes \mathbb{Z}_2$$

generated by the images of ψ, ι and ϕ_k for $k \geq 1$.

Remark 4.5. We remark that the relation $\iota\psi\iota = \psi^{-1}$ was omitted in Theorem 3.1 in [13].

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