CHAIN FLARING AND L^2–TORSION OF FREE-BY-CYCLIC GROUPS

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Abstract. We introduce a condition on the monodromy of a free-by-cyclic group, G_ϕ, called the *chain flare condition*, that implies that the L^2–torsion, $\rho^{(2)}(G_\phi)$, is non-zero. We conjecture that this condition holds whenever the monodromy is exponentially growing.

1. Introduction

The L^2–torsion, denoted $\rho^{(2)}(G)$, is an analytical group invariant that is well-defined for a large class of L^2–acyclic groups, i.e., a group whose L^2–homology vanishes. (For the remainder, when we speak of the L^2–torsion of an L^2–acyclic group G, we implicitly assume that G is in this class, which conjecturally includes all L^2–acyclic groups; see [25, Section 13] and [26, Section 13].) This invariant is a real number and it behaves similarly to Euler characteristic in the sense that it is multiplicative under covers and that there exists a sum formula along pushouts. If G is L^2–acyclic and G contains an elementary amenable normal subgroup, then it was shown by Wegner that $\rho^{(2)}(G) = 0$ [35], see also [25, Theorem 3.113]. In addition, there are a variety of conjectures and partial results relating the L^2–torsion of a group to the growth of torsion in homology; see the survey articles by Lück [26, Section 7] and [24, Section 3.6].

In the setting of the fundamental groups of 3–manifolds, the L^2–torsion was computed by Lück–Schick [27]. We recall their results in the context of an orientable 3–manifold that fibers over S^1 as this parallels the setting considered in this paper. Assuming for simplicity that the fiber is connected, such a 3–manifold is homeomorphic to a mapping torus:

$$M_f = \Sigma \times [0,1] \Big/ (x,0) \sim (f(x),1),$$

where $f\colon \Sigma \to \Sigma$ is a homeomorphism of an orientable connected surface Σ. Given such a homeomorphism $f\colon \Sigma \to \Sigma$, let $\mathcal{C}$ be the canonical cut system for f and assume for simplicity that each complementary component $\Sigma_s \subseteq \Sigma - \mathcal{C}$, $s = 1, \ldots, S$, is fixed by f. Thus the restriction of f to Σ_s determines a sub-mapping torus $M_{f,s} \subseteq M_f$. As $\mathcal{C}$ is the canonical cut system for f, the restriction of f to Σ_s is, up to homotopy, either periodic or pseudo-Anosov. When the restriction of f to Σ_s is pseudo-Anosov, Thurston proved that the manifold $M_{f,s}$ admits a complete hyperbolic metric [34]. The work of Lück–Schick in this setting shows that $-\rho^{(2)}(\pi_1(M_f))$ equals $\frac{1}{6\pi}$ times the sum—over the indices $1 \leq s \leq S$ where the restriction of f to Σ_s is pseudo-Anosov—of the volumes of these hyperbolic sub-mapping tori $M_{f,s}$. In particular, the L^2–torsion is determined by the exponential dynamics of f. Combining this with work of Gromov [18], Soma [32] and Thurston [33], this implies that the L^2–torsion $-\rho^{(2)}(\pi_1(M_f))$ is proportional to the simplicial volume $\|M_f\|$. Similarly, combining this with work of Pieroni [30] this also implies that the L^2–torsion is proportional to the cube of the minimal volume entropy $\omega(M_f)^3$.

Building onto the established research by Algom-Kfir–Hironaka–Rafi [1], Dowdall–Kapovich–Leininger [11, 12, 13], Funke–Kielak [17] and others of studying free-by-cyclic groups analogously to 3–manifolds that fiber over S^1, the aim of this paper is to study the L^2–torsion of a free-by-cyclic

group, in particular, trying to understand when this invariant is non-zero. A *free-by-cyclic group* is a group that fits into a short exact sequence:

$$1 \to \mathbb{F} \to G \to \mathbb{Z} \to 1$$

where $\mathbb{F}$ is a finitely generated free group and hence it admits a presentation as a semi-direct product:

$$G = \mathbb{F} \rtimes_\Phi \langle t \rangle = \langle \mathbb{F}, t \mid t^{-1}xt = \Phi(x) \text{ for } x \in \mathbb{F} \rangle$$

where $\Phi \in \mathrm{Aut}(\mathbb{F})$. Changing the automorphism Φ within its outer automorphism class amounts to replacing the generator t by tx for some $x \in \mathbb{F}$ and so we are justified in denoting the above defined group by G_ϕ where $\phi = [\Phi] \in \mathrm{Out}(\mathbb{F})$.

Previously, building off of work by Lück [25, Section 7.4], the author showed how to compute $-\rho^{(2)}(G_\phi)$ using a topological representative $f\colon \Gamma \to \Gamma$ of $\phi \in \mathrm{Out}(\mathbb{F})$ [9]. Similar to the setting of 3–manifolds that fiber over S^1 mentioned above, it was shown that $-\rho^{(2)}(G_\phi)$ can be expressed using a topological representative $f\colon \Gamma \to \Gamma$ for $\phi \in \mathrm{Out}(\mathbb{F})$. In this context, Γ is a graph, f is a homotopy equivalence and there is filtration $\emptyset = \Gamma_0 \subset \Gamma_1 \subset \cdots \subset \Gamma_S = \Gamma$ by subgraphs that is respected by f in the sense that $f(\Gamma_s) \subseteq \Gamma_s$ for each $s = 1, \ldots, S$. For each $1 \leq s \leq S$, there is a non-negative integer matrix $M(f)_s$ that records the number of times the image of an edge in $\Gamma_s - \Gamma_{s-1}$ crosses an edge in $\Gamma_s - \Gamma_{s-1}$. Enlarging the filtration if necessary, we can assume that each matrix $M(f)_s$ is either the zero matrix or it is irreducible. With this set-up $-\rho^{(2)}(G_\phi)$ is expressed as a sum over the the indices $1 \leq s \leq S$ where $M(f)_s$ is irreducible and has Perron–Frobenius eigenvalue strictly greater than 1; this subset of indices is denoted $\mathcal{EG}(f)$. Each term in the summation is the logarithm of the Fuglede–Kadison determinant of an operator associated to the restriction of f to the subgraph Γ_s. Hence, as mentioned previously in the setting of 3–manifolds that fiber over S^1, the L^2–torsion of a free-by-cyclic group is determined by the exponential dynamics of the monodromy f. While the simplicial volume of a free-by-cyclic group is not well-defined, the minimal volume entropy is. It was recently shown by Bregman and the author that the L^2–torsion $-\rho^{(2)}(G_\phi)$ is not proportional to the square of the minimal volume entropy $\omega(G_\phi)^2$ in general [6].

Lück has shown that $-\rho^{(2)}(G_\phi)$ is non-negative for any free-by-cyclic group [25, Theorem 7.29]. The main result in the author's previous work [9] provides an upper bound on $-\rho^{(2)}(G_\phi)$ in terms of the data previously described, namely the matrices $M(f)_s$. In particular, it was shown that $-\rho^{(2)}(G_\phi) = 0$ when ϕ is polynomially growing, i.e., $\mathcal{EG}(f) = \emptyset$. In this paper, we suggest a strategy to show that $-\rho^{(2)}(G_\phi) > 0$ whenever ϕ is exponentially growing, i.e., $\mathcal{EG}(f) \neq \emptyset$. To this end, we introduce a condition, called the chain flare condition, and prove that this implies $-\rho^{(2)}(G_\phi) > 0$.

Theorem 1.1. *Suppose that $f\colon \Gamma \to \Gamma$ is a homotopy equivalence that respects the reduced filtration $\emptyset = \Gamma_0 \subset \Gamma_1 \subset \cdots \subset \Gamma_S = \Gamma$ and that $f\colon \Gamma \to \Gamma$ represents the outer automorphism $\phi \in \mathrm{Out}(\mathbb{F})$. If the restriction of f to Γ'_s satisfies the chain flare condition relative to $\Gamma'_s \cap \Gamma_{s-1}$ for each $s \in \mathcal{EG}(f)$, then*

$$-\rho^{(2)}(G_\phi) = \sum_{s \in \mathcal{EG}(f)} \int_{1<|z|} \log|z| \, d\mu_{L_{f,s}}. \tag{1.1}$$

Moreover, each integral in (1.1) is positive and hence $-\rho^{(2)}(G_\phi) > 0$.

In Theorem 1.1, the filtration $\emptyset = \Gamma_0 \subset \Gamma_1 \subset \cdots \subset \Gamma_S = \Gamma$ being reduced means that there is a single component $\Gamma'_s \subseteq \Gamma_s$ that is not contained in Γ_{s-1}. The measures $\mu_{L_{f,s}}$ appearing in (1.1) are the Brown measures [8] associated to the operators $L_{f,s}\colon L^2(G_\phi)^{n_s} \to L^2(G_\phi)^{n_s}$ (n_s is the number

of edges in $\Gamma_s - \Gamma_{s-1}$). These operators are described fully in Section 3.3, but briefly, they are induced from the vertical flow in the universal cover of the mapping torus for f.

The chain flare condition is the linear analog of the annuli flare condition of Bestvina–Feighn [3] that has been successfully employed by Bestvina–Feighn–Handel [4] and Brinkmann [7] to prove hyperbolicity of certain free-by-cyclic groups, and more generally by Kapovich [22] and Mutanguha [28] to prove hyperbolicity of certain ascending HNN-extensions over free groups. Full details regarding the chain flare condition will be given in Section 2, but we provide a quick explanation here. We can lift the topological representative $f\colon \Gamma \to \Gamma$ of $\phi \in \mathrm{Out}(\mathbb{F})$ to a cellular map $\tilde{f}\colon \widetilde{\Gamma} \to \widetilde{\Gamma}$ where $\widetilde{\Gamma}$ is the universal cover of the graph Γ. This map is not $\mathbb{F}$–equivariant, but satisfies $\tilde{f}(gz) = \Phi_f \tilde{f}(z)$ where $g \in \mathbb{F}$, $z \in \widetilde{\Gamma}$ and $\Phi_f \in \mathrm{Aut}(\mathbb{F})$ represents ϕ. The map $\tilde{f}$ induces a map on the level of cellular 1–chains of $\widetilde{\Gamma}$, which we denote by $A_f\colon C_1(\widetilde{\Gamma};\mathbb{Q}) \to C_1(\widetilde{\Gamma};\mathbb{Q})$. In spirit, the chain flare condition asserts the existence of a constant $\lambda > 1$ such that for any 1–chain $x \in C_1(\widetilde{\Gamma};\mathbb{Q})$ we have:

$$\lambda \|A_f(x)\| \leq \max\left\{\left\|A_f^2(x)\right\|, \|x\|\right\}$$

—where $\|\blacksquare\|$ is the usual L^2–norm—unless the 1–chain x has an obvious reason why it should not satisfy this inequality, e.g., x is fixed by A_f. The previously mentioned work of Bestvina–Feighn–Handel [4] and Brinkmann [7] implies that the chain flare condition always holds when the support of the boundary of x consists of two points, i.e., x is the 1–chain determined by an edge-path in $\widetilde{\Gamma}$.

Before we give an outline of the proof of Theorem 1.1 and the rest of the paper, we mention some related results. For a free-by-cyclic group $\mathbb{F} \rtimes_{\Phi_f} \langle t \rangle$, the L^2–torsion is the logarithm of the Fuglede–Kadison determinant of the operator given by right multiplication by $I - tJ_1(f)$ where $J_1(f)$ is a matrix with entries in the group ring $\mathbb{Z}[\mathbb{F}]$, see Section 3.3. Deninger has computed the Fuglede–Kadison determinant of similar operators for the discrete Heisenberg group [10]. Specifically, the discrete Heisenberg groups can be expressed as a semi-direct product $\mathbb{Z}^2 \rtimes_\Phi \langle t \rangle$ where t acts on $\mathbb{Z}^2$ via the matrix $\left[\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right]$. Deninger studies operators given by right multiplication by $1 - xt$ where $x \in \mathbb{C}[\mathbb{Z}^2]$ and expresses the logarithm of the Fuglede–Kadison determinant as an integral of $\log|x|$, treating x as a polynomial in two variables [10, Theorem 11]. Funke–Kielak study a different variant of the L^2–torsion of free-by-cyclic groups, called the L^2–torsion polytope [17], see also the later work by Kielak [23]. Together, these works show that the BNS invariant of the free-by-cyclic group is determined by the L^2–torsion polytope. The connection between their work and the present work is the universal L^2–torsion defined by Friedl–Lück [16]. This is a certain element in the weak Whitehead group $\rho_u^{(2)}(G) \in \mathrm{Wh}^w(G)$ associated to the L^2–acyclic group G. The logarithm of the Fuglede–Kadison determinant gives a homomorphism $\mathrm{Wh}^w(G) \to \mathbb{R}$; the L^2–torsion $\rho^{(2)}(G)$ is the image of $\rho_u^{(2)}(G)$. There is another homomorphism defined on $\mathrm{Wh}^w(G)$ by Friedl–Lück [16] whose image is a polytope; the L^2–torsion polytope is the image of $\rho_u^{(2)}(G)$.

1.1. **Outline of the proof.** The proof of Theorem 1.1 builds off of the author's previous work [9]. To simplify the exposition in this introductory section, we will assume that the filtration consists of a single stratum, i.e., that $S = 1$. For notational simplicity, we will denote the operator $L_{f,1}$ simply by L and n_1 simply by n (the number of edges in Γ). Using the property that the L^2–torsion is multiplicative under covers and properties of the Brown measure, the proof of Theorem 1.1 starts by showing that for any $k \geq 1$, we can express the L^2–torsion as a certain integral over $\mathbb{C}$:

$$-\rho^{(2)}(G_\phi) = \frac{1}{k}\int_{\mathbb{C}} \log\left|1 - z^k\right| \, d\mu_L.$$

We would like to take the limit as $k \to \infty$. Notice that $\log \left|1 - z^k\right|^{1/k} \to 0$ as $k \to \infty$ when $|z| < 1$ and that $\log \left|1 - z^k\right|^{1/k} \to \log |z|$ as $k \to \infty$ when $1 < |z|$. On the unit circle, the limit does not exist. In order to apply the Lebesgue dominated convergence theorem, we need to bound the integrand and this results in bounding away from the unit circle. To this end, for $\nu > 1$, we separate the above integral into integrals over three regions: (i) $|z| < \nu^{-1}$, (ii) $\nu^{-1} \leq |z| \leq \nu$, and (iii) $\nu < |z|$. The integral over the first region thus limits to 0 and the integral over the third region limits to the integral of $\log |z|$ over $\nu < |z|$. It is the integral over the region $\nu^{-1} \leq |z| \leq \nu$ that requires further investigation. We desire to show that this integral is equal to 0. Using the work of Haagerup–Schultz [20] on the invariant subspace problem, this integral is equal to $\frac{1}{k}$ times the logarithm of the Fuglede–Kadison determinant of the operator $I - L^k$ restricted to a certain invariant subspace $\mathfrak{K}_\nu \subseteq L^2(G_\phi)^n$, see Theorem 4.4. This is where the chain flare condition comes into play. Under the chain flare condition we can identify this subspace for ν sufficiently close to and greater than 1. There are two cases:

(1) The subspace $\mathfrak{K}_\nu$ is the trivial subspace. In the language of the chain flare condition, this happens when there is no Nielsen 1–chain so that the quasi-fixed submodule V_{qf} is trivial. By definition, the 0 morphism has Fuglede–Kadison determinant equal to 1 and so the integral over the region $\nu^{-1} \leq |z| \leq \nu$ is 0 as desired.

(2) The subspace $\mathfrak{K}_\nu$ is isomorphic to $L^2(G_\phi)$ and the restriction of $I - L^k$ to this subspace is induced from the operator given by right multiplication by $1 - t^k$ on the subgroup $\langle t \rangle \subset G_\phi$. In the language of the chain flare condition, this happens when there is a Nielsen 1–chain that generates the quasi-fixed submodule V_{qf}. Hence, by properties of the Fuglede–Kadison determinant, the determinant of the restriction of $I - L^k$ to $\mathfrak{K}_\nu$ equals the determinant of the operator given by right multiplication by $1 - t^k$ on $L^2(\langle t \rangle)$. This operator has determinant equal to 1 and again so the integral over the region $\nu^{-1} \leq |z| \leq \nu$ is 0 as desired.

Since this holds for all $\nu > 1$, we conclude that:

$$-\rho^{(2)}(G_\phi) = \int_{1<|z|} \log |z| \; d\mu_L$$

as claimed.

To complete the proof of Theorem 1.1, we must show that the integral in the above equation is positive. Using the properties of the Brown measure and the operator L, we show in proof of Theorem 1.1 that:

$$0 \leq \int_{|z|<1} \log |z| \; d\mu_L + \int_{1<|z|} \log |z| \; d\mu_L$$

As the first integral is non-positive, the second integral is non-negative and as we can show that the support of the measure $d\mu_L$ is not contained in the unit circle, we conclude that the second integral is in fact positive.

1.2. **Organization of paper.** This paper is organized as follows. In Section 2 we introduce the concepts and notation necessary to state the chain flare condition, which is formally stated in Section 2.4. Sections 3.1 and 3.2 define the notion of the Fuglede–Kadison determinant and the L^2–torsion, especially in the context of free-by-cyclic groups. The author's previous work on computing this invariant using a topological representative, in particular the definition of the operators in Theorem 1.1 is recalled in Section 3.3. The Brown measure associated to Hilbert–G–module morphism $A\colon U \to U$ is introduced in Section 4.1 and its relation to the Haagerup–Schultz invariant subspaces is explained in Section 4.2. The work on using the chain flare condition to understand the invariant subspace $\mathfrak{K}_\nu$ is initiated in Section 5 where we explore the dynamics on

quasi-fixed submodule V_{qf} mentioned in Section 1.1. The are two cases, depending on whether the Nielsen 1–chain is non-geometric (Section 5.1) or geometric (Section 5.2). In Sections 6 and 7, we identify the subspace $\mathfrak{K}_\nu$ as explained in Section 1.1 and compute the Fuglede–Kadison determinant of the restriction of the operator $I - L_{f,s}$ to this subspace. The proof of Theorem 1.1 takes place in Section 8. We conclude in Section 9 with some final remarks regarding the chain flare condition and on applying the methods within to ascending HNN-extensions $\mathbb{F}*_\Psi$.

2. The chain flare condition

The goal of this section is to give the complete statement of the chain flare condition. There are several technicalities that are necessary to derive a statement that works for a general topological representative $f\colon \Gamma \to \Gamma$ and that takes into account invariant subgraphs. On a first read, the reader is invited in Sections 2.2 and 2.4 to assume that the graph Γ is a rose and that the invariant subgraph $H \subset \Gamma$ is a single vertex. In this case, all of the $\mathbb{Q}[\mathbb{F}]$–modules defined in Section 2.2 are equal and isomorphic to $\mathbb{Q}[\mathbb{F}]^n$ (n is the rank of $\mathbb{F}$) and the homomorphism $A_{f,H}$ is an isomorphism of this free module.

2.1. **Graphs and morphisms.** A *graph* is a 1–dimensional CW–complex. If Γ is a graph, by $\mathsf{V}(\Gamma)$ we denote the set of *vertices* (0–cells) and by $\mathsf{E}(\Gamma)$ we denote the set of *edges* (1–cells). As 1–cells, edges are oriented; the initial vertex of an edge $e \in \mathsf{E}(\Gamma)$ is denoted $\mathsf{o}(e)$ and the terminal vertex is denoted $\mathsf{t}(e)$. The same edge with opposite orientation is denoted by $\bar{e}$.

An *edge-path* is the image of a continuous map $\mathbf{p}\colon [0,1] \to \Gamma$ for which there exists a partition $0 = x_0 < x_1 < \cdots < x_m = 1$ such that $\mathbf{p}|_{[x_{k-1},x_k]}$ is homeomorphism onto an edge of Γ. When there is no ambiguity, we will define an edge-path by listing the vertices it visits and write $\mathbf{p}\colon p_0, \ldots, p_m$ where $p_k = \mathbf{p}(x_k)$ or by listing the edges it visits.

For a graph Γ, a *morphism* $f\colon \Gamma \to \Gamma$ is a cellular map that linearly expands each edge of Γ across an edge-path in Γ (with respect to some metric). Fixing an enumeration of edges of Γ, $\mathsf{E}(\Gamma) = \{e_1, \ldots, e_n\}$, the *transition matrix* $M(f)$ is the $n \times n$ matrix where $m_{i,j}$ equals the number of occurrences of e_j or $\bar{e}_j$ in the edge-path $f(e_i)$.

The morphism $f\colon \Gamma \to \Gamma$ *respects* a filtration of Γ by subgraphs $\emptyset = \Gamma_0 \subset \Gamma_1 \subset \cdots \subset \Gamma_S = \Gamma$ if $f(\Gamma_s) \subseteq \Gamma_s$ for all $1 \leq s \leq S$. In this case, the transition matrix can be assumed to have a lower block triangular form. Indeed this happens so long as edges lower in the filtration are ordered first, i.e, $e_i \in \mathsf{E}(\Gamma_s)$ and $e_j \notin \mathsf{E}(\Gamma_s)$ implies that $i < j$. Let i_s denote the smallest index with $e_{i_s} \notin \mathsf{E}(\Gamma_{s-1})$, let $n_s = \#\,|\mathsf{E}(\Gamma_s) - \mathsf{E}(\Gamma_{s-1})|$ and let $M(f)_s$ denote the $n_s \times n_s$ submatrix of $M(f)$ with $i_s \leq i, j \leq i_s + n_s - 1$. Then $M(f)$ is lower block triangular with the submatrices $M(f)_s$ along the diagonal.

Unless otherwise stated, we will always assume that such a filtration $\emptyset = \Gamma_0 \subset \Gamma_1 \subset \cdots \subset \Gamma_S = \Gamma$ is maximal in the sense that $M(f)_s$ is either the zero matrix or irreducible for each $1 \leq s \leq S$. For each $1 \leq s \leq S$ where $M(f)_s$ is irreducible, we let $\lambda(f)_s$ denote the associated Perron–Frobenius eigenvalue. We set $\mathcal{EG}(f) = \{s \mid M(f)_s \text{ is irreducible and } \lambda(f)_s > 1\}$. We say the filtration $\emptyset = \Gamma_0 \subset \Gamma_1 \subset \cdots \subset \Gamma_S = \Gamma$ is *reduced* if for each $1 \leq s \leq S$ there is exactly one component $\Gamma'_s \subseteq \Gamma_s$ that is not a component of Γ_{s-1}. (This definition is not the standard usage of the term reduced in the context of filtrations—cf. [5, 15, 21], but our definition is easily implied by the standard usage and our definition is what we use in the sequel.)

We say a homotopy equivalence $f\colon \Gamma \to \Gamma$ *represents* an outer automorphism $\phi \in \mathrm{Out}(\mathbb{F})$ if there is a vertex $* \in \mathsf{V}(\Gamma)$, an identification $\pi_1(\Gamma, *) \cong \mathbb{F}$ and an edge-path from $*$ to $f(*)$ such that the outer automorphism induced by f and this edge-path is ϕ. Unless otherwise noted, all maps of graphs in the sequel are morphisms.

2.2. **Relative 1–chains.** Suppose $f\colon \Gamma \to \Gamma$ is a homotopy equivalence that fixes a vertex $* \in \mathsf{V}(\Gamma)$ and that $H \subset \Gamma$ is an f–invariant subgraph. Fixing an isomorphism $\pi_1(\Gamma, *) \cong \mathbb{F}$ we have that f and the trivial path based at $*$ induces an automorphism of $\mathbb{F}$ that we denote by Φ_f.

Let $\widetilde{\Gamma}$ be the universal cover of Γ and let $\widetilde{H}$ be the union of the lifts of H to $\widetilde{\Gamma}$. Fix a lift $\tilde{*}$ of $*$ to $\widetilde{\Gamma}$ and let $\tilde{f}$ be the lift of f such that $\tilde{f}(\tilde{*}) = \tilde{*}$. This map satisfies $\tilde{f}(gz) = \Phi_f(g)\tilde{f}(z)$ for any point $z \in \widetilde{\Gamma}$ and any element $g \in \mathbb{F}$. The set of rational (cellular) 1–chains, $C_1(\widetilde{\Gamma}; \mathbb{Q})$, is a $\mathbb{Q}[\mathbb{F}]$–module isomorphic to $\mathbb{Q}[\mathbb{F}]^n$, where n is the number of edges in Γ. We express 1–chains as formal linear combinations of the edges in $\widetilde{\Gamma}$ and write $x = \sum_{e \in \mathsf{E}(\widetilde{\Gamma})} x_e e$ where $x_e \in \mathbb{Q}$ and $x_e \neq 0$ for only finitely many $e \in \mathsf{E}(\Gamma)$. The *support* of a 1–chain $x \in C_1(\widetilde{\Gamma}; \mathbb{Q})$ is defined by $\operatorname{supp}(x) = \{e \in \mathsf{E}(\widetilde{\Gamma}) \mid x_e \neq 0\}$. Note that changing the orientation on e swaps the sign of the corresponding coefficient. The map $\tilde{f}$ induces an abelian group homomorphism $A_f\colon C_1(\widetilde{\Gamma}; \mathbb{Q}) \to C_1(\widetilde{\Gamma}; \mathbb{Q})$ that satisfies $A_f(gx) = \Phi_f(g)A_f(x)$ for any 1–chain $x \in C_1(\widetilde{\Gamma}; \mathbb{Q})$ and any element $g \in \mathbb{F}$.

Similarly, we also consider the of rational (cellular) 0–chains $C_0(\widetilde{\Gamma}; \mathbb{Q})$ and the usual boundary map $\partial_1\colon C_1(\widetilde{\Gamma}; \mathbb{Q}) \to C_0(\widetilde{\Gamma}; \mathbb{Q})$ defined on edges $\partial_1 e = \mathsf{t}(e) - \mathsf{o}(e)$.

Given distinct vertices $u_1, u_2 \in \mathsf{V}(\widetilde{\Gamma})$, by $[u_1, u_2]$ we denote the 1–chain in $C_1(\widetilde{\Gamma}; \mathbb{Q})$ uniquely determined by:

$$\partial_1[u_1, u_2]_v = \begin{cases} -1 & \text{if } v = u_1 \\ 1 & \text{if } v = u_2 \\ 0 & \text{else.} \end{cases}$$

In particular, $[u_1, u_2]_e = \pm 1$ for any edge in the edge-path from u_1 to u_2 and $[u_1, u_2]_e = 0$ for all other edges.

We consider the following $\mathbb{Q}[\mathbb{F}]$–submodule of rational 1–chains in $\widetilde{\Gamma}$ relative to $\widetilde{H}$:

$$C_1(\widetilde{\Gamma}, \widetilde{H}; \mathbb{Q}) = \{x \in C_1(\widetilde{\Gamma}; \mathbb{Q}) \mid x_e = 0 \ \forall e \in \mathsf{E}(\widetilde{H})\}.$$

We observe that $C_1(\widetilde{\Gamma}; \mathbb{Q}) = C_1(\widetilde{H}; \mathbb{Q}) \oplus C_1(\widetilde{\Gamma}, \widetilde{H}; \mathbb{Q})$. By π_H and $\pi_H^\perp$ respectively we denote the projections of $C_1(\widetilde{\Gamma}; \mathbb{Q})$ onto $C_1(\widetilde{H}; \mathbb{Q})$ and $C_1(\widetilde{\Gamma}; \widetilde{H}; \mathbb{Q})$ respectively. The following homomorphism is central to the chain flare condition:

$$A_{f,H} = \pi_H^\perp \circ A_f\big|_{C_1(\widetilde{\Gamma}, \widetilde{H}; \mathbb{Q})}\colon C_1(\widetilde{\Gamma}, \widetilde{H}; \mathbb{Q}) \to C_1(\widetilde{\Gamma}, \widetilde{H}; \mathbb{Q}).$$

2.3. **Nielsen 1–chains.** As stated in the Introduction, in essence, the chain flare condition states that the norm of a relative 1–chain in $C_1(\widetilde{\Gamma}, \widetilde{H}; \mathbb{Q})$ should grow by a definite factor after applying $A_{f,H}$ or else it is the image of a relative 1–chain whose norm is a larger by a definite factor. However, there are certain 1–chains that are fixed by $A_{f,H}$ that need to be accounted for. This is the motivation for the definition of a Nielsen 1–chain.

Definition 2.1. Let $\rho \in C_1(\widetilde{\Gamma}, \widetilde{H}; \mathbb{Q})$ be a relative 1–chain such that $\rho = \pi_H^\perp([u, v])$ for some vertices $u, v \in \mathsf{V}(\widetilde{\Gamma})$ that are fixed by $\tilde{f}$, i.e., $\tilde{f}(u) = u$ and $\tilde{f}(v) = v$. We say ρ is a *non-geometric Nielsen 1–chain* if it satisfies the following condition.

(NNC1) There is an edge $e \in \mathsf{E}(\widetilde{\Gamma}) - \mathsf{E}(\widetilde{H})$ such that $\rho_e = \pm 1$ and $\rho_{ge} = 0$ for any non-trivial element $g \in \mathbb{F}$.

We say ρ is a *geometric Nielsen 1–chain* if it satisfies the following conditions.

(GNC1) For distinct elements $g_1, g_2 \in \mathbb{F}$, the intersection $\operatorname{supp}(g_1\rho) \cap \operatorname{supp}(g_2\rho)$ is either empty or consists of a single edge.

(GNC2) For all edges $e \in \mathsf{E}(\widetilde{\Gamma}) - \mathsf{E}(\widetilde{H})$, there are exactly two elements $g_1, g_2 \in \mathbb{F}$ such that e is the unique edge in the intersection $\mathrm{supp}(g_1\rho) \cap \mathrm{supp}(g_2\rho)$.

(GNC3) There exists non-commuting elements $g_1, g_2 \in \mathbb{F}$ such that the intersections $\mathrm{supp}(\rho) \cap \mathrm{supp}(g_1\rho)$ and $\mathrm{supp}(\rho) \cap \mathrm{supp}(g_2\rho)$ are non-empty.

We say ρ is a *Nielsen 1–chain* if it is either a non-geometric or a geometric Nielsen 1–chain.

We observe that $A_{f,H}(\rho) = \rho$. In Section 9 we explain how Nielsen 1–chains naturally arise for EG strata in a CT map.

2.4. **The chain flare condition.** We require the following notation before we state the chain flare condition. We consider the usual L^2–inner product and L^2–norm on 1–chains. That is, given a 1–chains $x = \sum_{e\in\mathsf{E}(\widetilde{\Gamma})} x_e e$ and $x' = \sum_{e\in\mathsf{E}(\widetilde{\Gamma})} x'_e e$ we set:

$$\langle x, x' \rangle = \sum_{e\in\mathsf{E}(\widetilde{\Gamma})} x_e x'_e \text{ and } \|x\|^2 = \langle x, x\rangle = \sum_{e\in\mathsf{E}(\widetilde{\Gamma})} |x_e|^2 .$$

If $V \subseteq C_1(\widetilde{\Gamma}; \widetilde{H}; \mathbb{Q})$ is a $\mathbb{Q}[\mathbb{F}]$–submodule and $0 < \theta < 1$, we set:

$$N_\theta(V) = \left\{ x' \in C_1(\widetilde{\Gamma}, \widetilde{H}; \mathbb{Q}) \mid \langle x, x' \rangle > \theta \|x\| \left\|x'\right\| \text{ for some } x \in V \right\}.$$

Thus, $N_\theta(V)$ consists of elements that make a small angle with an element of V. Notice that $N_\theta(V) - \{0\}$ is a neighborhood of $V - \{0\}$.

If $N \subseteq C_1(\widetilde{\Gamma}, \widetilde{H}; \mathbb{Q})$ is a subset, we define the following subset:

$$N^\infty = \bigcup_{k\in\mathbb{Z}} A^k_{f,H}(N).$$

In other words, N^∞ consists of all relative 1–chains x' such that either $x' = A^k_{f,H}(x)$ for some $x \in V$ and some $k \geq 0$ or that $A^k_{f,H}(x') \in V$ for some $k \geq 0$. We remark that N^∞ is $A_{f,H}$–invariant.

We can now formally state the chain flare condition.

Chain Flare Condition. Suppose $f\colon \Gamma \to \Gamma$ is a homotopy equivalence and $H \subset \Gamma$ is an f–invariant subgraph. We say f *satisfies the chain flare condition relative to* H if there are $\mathbb{Q}[\mathbb{F}]$–submodules $V_{\mathrm{h}}, V_{\mathrm{qf}} \subseteq C_1(\widetilde{\Gamma}, \widetilde{H}; \mathbb{Q})$ where the following conditions hold.

(CFH1) $C_1(\widetilde{\Gamma}, \widetilde{H}; \mathbb{Q}) = V_{\mathrm{h}} + V_{\mathrm{qf}}$.

(CFH2) There exists constants $\lambda > 1$ and $0 < \theta < 1$ such that for all $x \in N_\theta(V_{\mathrm{h}})^\infty$:

$$\lambda \|A_{f,H}(x)\| \leq \max\left\{ \left\|A^2_{f,H}(x)\right\|, \|x\| \right\}.$$

(CFH3) If $V_{\mathrm{qf}} \neq \{0\}$, then there exists a Nielsen 1–chain $\rho \in C_1(\widetilde{\Gamma}, \widetilde{H}; \mathbb{Q})$ such that for all $x \in V_{\mathrm{qf}}$, there exist rational numbers $q_1, \ldots, q_r \in \mathbb{Q}$ and elements $g_1, \ldots, g_r \in \mathbb{F}$ such that $x = q_1 g_1 \rho + \cdots + q_r g_r \rho$.

We call V_{h} the *hyperbolic submodule* and V_{qf} the *quasi-fixed submodule*. If $V_{\mathrm{qf}} \neq \{0\}$, we say the Nielsen 1–chain ρ specified in (CFH3) *generates* the submodule. We remark that to verify (CFH2), one may assume that the coefficients in x are integral. Further, if $V_{\mathrm{qf}} = \{0\}$, then it suffices to verify (CFH2) only for $x \in V_{\mathrm{h}}$ as $N_\theta(V_{\mathrm{h}})^\infty$ equals V_{h} in this case. Moreover, if $C_1(\widetilde{\Gamma}, \widetilde{H}; \mathbb{Q}) = V_{\mathrm{h}} \oplus V_{\mathrm{qf}}$ and V_{h} is $A_{f,H}$–invariant, then it suffices to verify (CFH2) only for $x \in V_{\mathrm{h}}$. See Remark 6.4.

For use later on in Section 6, we record the following consequence of (CFH2).

Lemma 2.2. *Suppose that the homotopy equivalence $f\colon \Gamma \to \Gamma$ satisfies the chain flare condition relative to the f–invariant graph $H \subset \Gamma$ with constants λ and θ. The following statements hold.*

(1) *If* $x \in N_\theta(V_\mathrm{h})^\infty$, $j \geq 1$ *and* $\lambda \left\| A^j_{f,H}(x) \right\| \leq \left\| A^{j-1}_{f,H}(x) \right\|$, *then* $\lambda^j \left\| A^j_{f,H}(x) \right\| \leq \|x\|$.

(2) *If* $x \in N_\theta(V_\mathrm{h})^\infty$, $j \geq 1$ *and* $\lambda \|x\| \leq \|A_{f,H}(x)\|$, *then* $\lambda^j \|x\| \leq \left\| A^j_{f,H}(x) \right\|$.

(3) *If* $x \in N_\theta(V_\mathrm{h})^\infty$ *and* $N \geq 1$, *then:*

$$\lambda^N \left\| A^N_{f,H}(x) \right\| \leq \max \left\{ \left\| A^{2N}_{f,H}(x) \right\|, \|x\| \right\}.$$

Proof. To simplify notation, we denote $A_{f,H}$ by A in the proof.

We first prove (1) by induction. The statement is tautological for $j = 1$. Now suppose that $j \geq 2$, that (1) holds for $j-1$, that $x \in N_\theta(V_\mathrm{h})^\infty$ and that $\lambda \left\| A^j(x) \right\| \leq \left\| A^{j-1}(x) \right\|$. Since $A^{j-2}(x) \in N_\theta(V_\mathrm{h})^\infty$, by (CFH2) we must have:

$$\lambda \left\| A^{j-1}(x) \right\| \leq \max \left\{ \left\| A^j(x) \right\|, \left\| A^{j-2}(x) \right\| \right\}.$$

As $\lambda \left\| A^j(x) \right\| \leq \left\| A^{j-1}(x) \right\|$ by assumption, we must have $\lambda \left\| A^{j-1}(x) \right\| \leq \left\| A^{j-2}(x) \right\|$ as $\lambda > 1$. Hence, by induction $\lambda^{j-1} \left\| A^{j-1}(x) \right\| \leq \|x\|$. Therefore:

$$\lambda^j \left\| A^j(x) \right\| = \lambda^{j-1} \left(\lambda \left\| A^j(x) \right\| \right) \leq \lambda^{j-1} \left\| A^{j-1}(x) \right\| \leq \|x\|.$$

The proof of (2) is similar. We provide the details for completeness. Again, the statement is tautological for $j = 1$. Suppose that $j \geq 2$, that (2) holds for $j-1$, that $x \in N_\theta(V_\mathrm{h})^\infty$ and that $\lambda \|x\| \leq \|A(x)\|$. Since $A(x) \in N_\theta(V_\mathrm{h})^\infty$, by (CFH2) we must have:

$$\lambda \|A(x)\| \leq \max \left\{ \left\| A^2(x) \right\|, \|x\| \right\}.$$

Thus, as before we find that $\lambda \|A(x)\| \leq \left\| A^2(x) \right\|$. Hence, by induction $\lambda^{j-1} \|A(x)\| \leq \left\| A^{j-1} A(x) \right\| = \left\| A^j(x) \right\|$. Therefore:

$$\lambda^j \|x\| = \lambda^{j-1} (\lambda \|x\|) \leq \lambda^{j-1} \|A(x)\| \leq \left\| A^j(x) \right\|. \qquad \square$$

We now consider (3). By (CFH2), we have that $\lambda \left\| A^N(x) \right\| \leq \max \left\{ \left\| A^{N+1}(x) \right\|, \left\| A^{N-1}(x) \right\| \right\}$. If $\lambda \left\| A^N(x) \right\| \leq \left\| A^{N+1}(x) \right\|$, then by (2) we have $\lambda^N \left\| A^N(x) \right\| \leq \left\| A^{2N}(x) \right\|$. If $\lambda \left\| A^N(x) \right\| \leq \left\| A^{N-1}(x) \right\|$, then by (1) we have $\lambda^N \left\| A^N(x) \right\| \leq \|x\|$. This proves (3).

3. L^2–torsion of free-by-cyclic groups

In this section, we recall the definition of L^2–torsion of a free-by-cyclic group as well as some results necessary for the sequel. General references for the material in this section are the survey paper by Eckmann [14] and the book by Lück [25].

3.1. **The von Neumann algebra of a countable group.** Let G be a countable group. By $L^2(G)$ we denote the vector space of square summable functions $\xi \colon G \to \mathbb{C}$. We will express an element of $L^2(G)$ as a formal linear combination $\xi = \sum_{g \in G} \xi_g g$ where $\xi_g \in \mathbb{C}$ and $\sum_{g \in G} |\xi_g|^2 < \infty$. This is a Hilbert space with inner product:

$$\langle \xi, \xi' \rangle = \sum_{g \in G} \xi_g \overline{\xi'_g}.$$

The associated norm is denoted $\|\xi\| = \langle \xi, \xi \rangle^{1/2}$. The dense subspace of finitely supported functions is isomorphic (as a vector space) to the group algebra $\mathbb{C}[G]$ and as such we consider $\mathbb{C}[G]$ as a subspace of $L^2(G)$. The group G acts isometrically on both the left and the right of $L^2(G)$ where for $h \in G$ and $\xi \in L^2(G)$ we define:

$$h \cdot \xi = \sum_{g \in G} \xi_g hg = \sum_{g \in G} \xi_{h^{-1}g} g \text{ and } \xi \cdot h = \sum_{g \in G} \xi_g gh = \sum_{g \in G} \xi_{gh^{-1}} g.$$

By linearity, these extend to actions on $L^2(G)$ of the group algebra $\mathbb{C}[G]$ by bounded operators. In the sequel, when we say that some function or object related to $L^2(G)$ is G–equivariant or G–invariant, we are referring to the left action.

The *von Neumann algebra* of G, denoted $\mathcal{N}(G)$, is the algebra of G–equivariant bounded operators on $L^2(G)$. That is, an element $A \in \mathcal{N}(G)$ is a bounded operator $A\colon L^2(G) \to L^2(G)$ such that $A(g \cdot \xi) = g \cdot A(\xi)$ for all $g \in G$ and $\xi \in L^2(G)$. In particular, for each $x \in \mathbb{C}[G]$, the operator $A_x\colon L^2(G) \to L^2(G)$ defined by $A_x(\xi) = \xi \cdot x$ is G–equivariant and hence we can consider $\mathbb{C}[G]$ as a subalgebra of $\mathcal{N}(G)$.

There is a notion of *trace* for elements in $\mathcal{N}(G)$ that is defined by:

$$\mathrm{tr}_G(A) = \langle A(\mathrm{id}_G), \mathrm{id}_G \rangle$$

where id_G is the identity element of G. More generally, a G–equivariant bounded operator $A\colon L^2(G)^n \to L^2(G)^n$ can be expressed as a matrix $A = [A_{i,j}]$ where each $A_{i,j} \in \mathcal{N}(G)$ and we define:

$$\mathrm{tr}_G(A) = \sum_{i=1}^{n} \mathrm{tr}_G(A_{i,i}).$$

A *(finitely generated) Hilbert–G–module* is a Hilbert space U that admits an isometric action by G and for which there exists a G–equivariant isometric embedding $U \to L^2(G)^n$ for some n. The notion of trace allows for the definition of dimension of a Hilbert–G–module by:

$$\dim_G(U) = \mathrm{tr}_G(P_U)$$

where $P_U\colon L^2(G)^n \to L^2(G)^n$ is the projection onto the image of U.

A *morphism* of Hilbert–G–modules U and V is a G–equivariant bounded operator $A\colon U \to V$. For a morphism $A\colon U \to V$ of Hilbert–G–modules, we denote by $F_A\colon [0,\infty) \to [0,\infty)$ the *spectral density function* of A, that is, $F_A(\lambda) = \mathrm{tr}_G\left(E^{A^*A}_{\lambda^2}\right)$ where $\left\{E^{A^*A}_{\lambda}\right\}$ is the spectral family of A^*A. The *Fuglede–Kadison determinant of A* is defined by:

$$\det_G(A) = \exp \int_{0^+}^{\infty} \log(\lambda)\, dF_A$$

if the integral exists, and $\det_G(A)$ is defined to be 0 otherwise.

We record the following property of the Fuglede–Kadison determinant for later use.

Lemma 3.1. *Let $A\colon U \to U$ and $O\colon U \to V$ be morphisms of finite dimensional Hilbert–G–modules where A is injective and O is an isomorphism. Then:*

$$\det_G(A) = \det_G(OAO^{-1}).$$

Proof. Let I denote the identity operator $I\colon U \to U$. By [25, Theorem 3.14 (1)] we have that $1 = \det_G(I) = \det_G(OO^{-1}) = \det_G(O) \cdot \det_G(O^{-1})$. Hence, by [25, Theorem 3.14 (1)] again, we find:

$$\det_G(OAO^{-1}) = \det_G(O) \cdot \det_G(A) \cdot \det_G(O^{-1}) = \det_G(A). \qquad \square$$

3.2. **L^2–torsion of free-by-cyclic groups.** Let G be a countable group and let X be a CW–complex that admits a continuous action by G that freely permutes the cells of X and such that there are only finitely many G–orbits of cells. The cellular chain complex $C_*(X) = \{\partial_j\colon C_j(X) \to C_{j-1}(X)\}$ consists of free $\mathbb{Z}[G]$–modules of finite rank. Thus $C^{(2)}_*(X) = L^2(G) \otimes_{\mathbb{Z}[G]} C_*(X)$ is a

chain complex of Hilbert–G–modules. If $C_*^{(2)}(X)$ is weakly acyclic, i.e., $\ker \partial_j = \operatorname{clos}(\operatorname{im} \partial_{j+1})$ for all j, and $\det_G(\partial_j) \neq 0$ for all j, then the L^2–*torsion* of X is defined by:

$$\rho^{(2)}(X) = -\sum_{j \geq 0} (-1)^j \log \det_G(\partial_j).$$

The situation we are most often interested in is when G has a finite classifying space, BG, and $X = EG$. There is a large class of groups for which the chain complex of Hilbert–G–modules $\{\partial_j \colon C_j^{(2)}(EG) \to C_{j-1}^{(2)}(EG)\}$ satisfies the above assumptions and moreover the L^2–torsion of EG depends only on G and not the particular choice of BG [25, Lemma 13.6]. This class of groups includes free-by-cyclic groups in particular and thus we are justified in defining

$$\rho^{(2)}(G_\phi) = \rho^{(2)}(EG_\phi).$$

This invariant behaves in some ways like Euler characteristic. The following property illustrates this connection and is used later on.

Theorem 3.2 ([25, Theorem 7.27 (4)])**.** *Suppose ϕ is an outer automorphism of $\mathbb{F}$. Then for all $k \geq 1$:*

$$\rho^{(2)}(G_{\phi^k}) = k\rho^{(2)}(G_\phi).$$

3.3. **Computing the L^2–torsion from a topological representative.** In the remainder of this section, we briefly explain how to compute the L^2–torsion $-\rho^{(2)}(G_\phi)$ from a homotopy equivalence $f \colon \Gamma \to \Gamma$ that represents $\phi \in \operatorname{Out}(\mathbb{F})$. See [9, Section 4] for complete details. As in Section 2.2, we assume that f fixes a vertex $* \in \mathsf{V}(\Gamma)$ and fixing an isomorphism $\pi_1(\Gamma, *) \cong \mathbb{F}$, we let Φ_f denote the automorphism induced by f and the trivial path based at $*$. We will use the semi-direct product presentation $\mathbb{F} \rtimes_{\Phi_f} \langle t \rangle$ for the corresponding free-by-cyclic group G_ϕ. Let $\tilde{f} \colon \widetilde{\Gamma} \to \widetilde{\Gamma}$ be the corresponding lift of f to the universal cover and let $A_f \colon C_1(\widetilde{\Gamma}; \mathbb{Q}) \to C_1(\widetilde{\Gamma}; \mathbb{Q})$ be the corresponding homomorphism.

Let X_f be the mapping torus of f, that is:

$$X_f = \Gamma \times [0,1] \Big/ (x, 0) \sim (f(x), 1),$$

and let $\widetilde{X}_f$ the universal cover of X_f. An edge in $\widetilde{X}_f$ is called *horizontal* if it is the lift of an edge in $\Gamma \times \{0\} \subset X_f$ and *vertical* otherwise. The subspace of (cellular) 1–chains $C_1(\widetilde{X}_f)$ spanned by horizontal edges is a free $\mathbb{Z}[G_\phi]$–module of rank $n = \#\,|\mathsf{E}(\Gamma)|$. Likewise, the set of (cellular) 2–chains $C_2(\widetilde{X}_f)$ is also a free $\mathbb{Z}[G_\phi]$–module of rank $n = \#\,|\mathsf{E}(\Gamma)|$. Hence, after choosing appropriate bases, the cellular boundary map $\partial_2 \colon C_2(\widetilde{X}_f) \to C_1(\widetilde{X}_f)$ followed by projection to the subspace spanned by the horizontal edges determines a $\mathbb{Z}[G_\phi]$–module homomorphism:

$$\partial_{\mathrm{hor}} \colon \mathbb{Z}[G_\phi]^n \to \mathbb{Z}[G_\phi]^n$$

that is given by right multiplication by a matrix of the form $I - tJ_1(f)$ where I is the identity matrix and $J_1(f) \in \operatorname{Mat}_n(\mathbb{Z}[\mathbb{F}])$, which is the so-called *Jacobian*. Indeed, each 2–cell in $\widetilde{X}_f$ has a unique bottom edge e and the top edges are none other than $t\tilde{f}(e)$ so that the horizontal components of the boundary of this 2–cell is $e - t\tilde{f}(e)$. See Figure 1. The rows of $J_1(f)$ just record the edges in $A_f(e)$, using the isomorphism between $C_1(\widetilde{\Gamma})$ and $\mathbb{Z}[\mathbb{F}]^n$. Let $L_f \colon L^2(G_\phi)^n \to L^2(G_\phi)^n$ be the operator given by right multiplication by $tJ_1(f)$ so that $\partial_{\mathrm{hor}} = I - L_f$ where I is now consider as the identity operator. Then, if the image of every vertex in Γ is fixed by f, it was shown that (cf. [25, Theorem 7.29]):

$$-\rho^{(2)}(G_\phi) = \log \det_{G_\phi}(I - L_f).$$

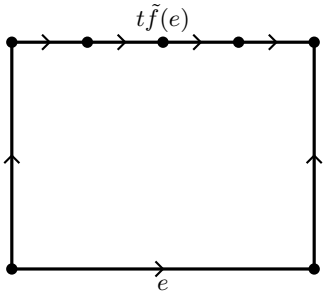

FIGURE 1. A 2–cell in $\widetilde{X}_f$.

When $f\colon \Gamma \to \Gamma$ respects a filtration $\emptyset = \Gamma_0 \subset \Gamma_1 \subset \cdots \subset \Gamma_S = \Gamma$ we can break up the above formula into pieces associated to the filtration elements. To this end, for each $1 \leq s \leq S$, we consider the subcomplex $X_{f,s} \subseteq X_f$ which is the mapping torus of the restriction of f to Γ_s. We note that this subcomplex is not necessarily connected. However, when the filtration is reduced, then there is exactly one component of $X_{f,s}$ that is not a component of $X_{f,s-1}$. Let $\widetilde{X}_{f,s}$ be the union of the lifts of $X_{f,s}$ to $\widetilde{X}_f$. The subspace of $C_1(\widetilde{X}_f)$ spanned by the horizontal edges that lie in $\widetilde{X}_{f,s} - \widetilde{X}_{f,s-1}$ is a free $\mathbb{Z}[G_\phi]$–module of rank $n_s = \#\,|\mathsf{E}(\Gamma_s) - \mathsf{E}(\Gamma_{s-1})|$. Hence as above, after choosing appropriate bases, the cellular boundary map $\partial_2\colon C_2(\widetilde{X}_{f,s} - \widetilde{X}_{f,s-1}) \to C_1(\widetilde{X}_{f,s})$ followed by projection to the subspace spanned by horizontal edges in $\widetilde{X}_{f,s} - \widetilde{X}_{f,s-1}$ determines a $\mathbb{Z}[G_\phi]$–module homomorphism:

$$\partial_{\mathrm{hor},s}\colon \mathbb{Z}[G_\phi]^{n_s} \to \mathbb{Z}[G_\phi]^{n_s}$$

that is given by right multiplication by a matrix of the form $I - tJ_1(f)_s$ where $J_1(f)_s \in \mathrm{Mat}_{n_s}(\mathbb{Z}[\mathbb{F}])$. In terms of matrices, $J_1(f)$ is lower block triangular with blocks $J_1(f)_s$ on the diagonal, as was the connection between $M(f)$ and $M(f)_s$. Let $L_{f,s}\colon L^2(G_\phi)^{n_s} \to L^2(G_\phi)^{n_s}$ by the operator given by right multiplication by $tJ_1(f)_s$ so that $\partial_{\mathrm{hor},s} = I - L_{f,s}$. Using the lower block triangular form of $J_1(f)$, which comes the lower block triangular form of $M(f)$, the following theorem was shown.

Theorem 3.3 ([9, Theorem 4.10 & Remark 4.12])**.** *Suppose that $f\colon \Gamma \to \Gamma$ is a homotopy equivalence that respects the filtration $\emptyset = \Gamma_0 \subset \Gamma_1 \subset \cdots \subset \Gamma_S = \Gamma$, that $f\colon \Gamma \to \Gamma$ represents the outer automorphism $\phi \in \mathrm{Out}(\mathbb{F})$ and that the image of each vertex in Γ is fixed by f. Then:*

$$-\rho^{(2)}(G_\phi) = \sum_{s \in \mathcal{EG}(f)} \log \det_{G_\phi}\big(I - L_{f,\,s}\big).$$

In the context of the chain flare condition we will use the following notation. Let $f\colon \Gamma \to \Gamma$ be a homotopy equivalence that represents $\phi \in \mathrm{Out}(\mathbb{F})$ and $H \subset \Gamma$ an f–invariant subgraph. We consider the filtration (which is not necessarily maximal nor reduced) $\emptyset = \Gamma_0 \subset \Gamma_1 \subset \Gamma_2 = \Gamma$ where $\Gamma_1 = H$. Then we set n_H to be equal to n_2, the number of edges in $\Gamma - H$, and set $L_{f,H}\colon L^2(G_\phi)^{n_H} \to L^2(G_\phi)^{n_H}$ to the be operator $L_{f,2}$.

4. Brown measure and Haagerup–Schultz invariant subspaces

In this section we introduce the Brown measure μ_A for a G–equivariant bounded operator $A\colon L^2(G)^n \to L^2(G)^n$, state its relation to the Fuglede–Kadison determinant and list the key properties that we require for the sequel. Additionally, we introduce the Haagerup–Schultz invariant subspaces $\mathfrak{E}(A,\nu)$ and $\mathfrak{F}(A,\nu)$ associated to bounded operator on $A\colon L^2(G)^n \to L(G)^n$

and state their relation to the Brown measure in the previously mentioned case when A is G–equivariant. The most important result of this section is Theorem 4.4 which is essential for the proof of Theorem 1.1. The results in this section hold for more general von Neumann algebras but are stated in setting in which they will be applied within.

4.1. **Brown measure.** Let G be a countable group and U a Hilbert–G–module. Associated to a morphism $A\colon U \to U$ is a Borel measure on $\mathbb{C}$, called the *Brown measure* and denoted μ_A [8]. This measure maybe considered as giving the multiplicity of the values of the spectrum of A. Indeed, if G is a finite group, then U is isomorphic as a vector space to $\mathbb{C}^{n|G|}$ for some n and considering A as an element of $\mathrm{Mat}_{n|G|}(\mathbb{C})$ we have:

$$\mu_A = \frac{1}{|G|}\sum_{j=1}^{n|G|}\delta_{\lambda_j}$$

where $\lambda_1, \ldots, \lambda_{n|G|}$ are the eigenvalues of A listed with multiplicity and δ_λ is the Dirac measure concentrated on the complex number λ.

We summarize some of the properties of the Brown measure and its relation to the Fuglede–Kadison in the following theorem.

Theorem 4.1 ([8, Theorem 3.13])**.** *Let $A\colon U \to U$ be a morphism of Hilbert–G–modules. The following properties hold.*

(1) The support of μ_A is contained in the spectrum of A.
(2) $\mu_A(\mathbb{C}) = \dim_G(U)$.
(3) If $h\colon \mathbb{C} \to \mathbb{C}$ is holomorphic, then

$$\log\det{}_G(h(A)) = \int_{\mathbb{C}} \log|h(z)|\; d\mu_A.$$

4.2. **Haagerup–Schultz invariant subspaces.** In their study of the invariant subspace problem for operators in a type Π_1–factor, Haagerup–Schultz identified the following subspaces associated to a bounded operator on a Hilbert space.

Definition 4.2 ([20, Definition 3.1 & Lemma 3.2])**.** Let $A\colon \mathcal{H} \to \mathcal{H}$ be a bounded operator on a Hilbert space. For $\nu > 0$ we define the following A–invariant closed subspaces of $\mathcal{H}$:

$$\mathfrak{E}(A,\nu) = \left\{\xi \in \mathcal{H} \mid \exists(\xi_j) \subset \mathcal{H} \text{ with } \lim_{j\to\infty}\|\xi_j - \xi\| = 0 \text{ and } \limsup_{j\to\infty}\left\|A^j\xi_j\right\|^{1/j} \leq \nu\right\}$$

$$\mathfrak{F}(A,\nu) = \left\{\xi \in \mathcal{H} \mid \exists(\xi_j) \subset \mathcal{H} \text{ with } \lim_{j\to\infty}\left\|A^j\xi_j - \xi\right\| = 0 \text{ and } \limsup\|\xi_j\|^{1/j} \leq \nu^{-1}\right\}$$

Remark 4.3 ([20, Remark 3.3])**.** If $A\colon \mathcal{H} \to \mathcal{H}$ is invertible, then:

$$\mathfrak{F}(A,\nu) = \mathfrak{E}(A^{-1},\nu^{-1}) = \left\{\xi \in \mathcal{H} \mid \exists(\xi_j) \subset \mathcal{H} \text{ with } \lim_{j\to\infty}\|\xi_j - \xi\| = 0 \text{ and } \limsup_{j\to\infty}\left\|A^{-j}\xi_j\right\|^{1/j} \leq \nu^{-1}\right\}$$

There is a deep connection between these subspaces and the Brown measure μ_A when $A\colon U \to U$ is a morphism of Hilbert–G–modules. In particular, Haagerup–Schultz prove that the dimension of $\mathfrak{E}(A,\nu)$ is the Brown measure of $\{z \in \mathbb{C} \mid |z| \leq \nu\}$ and the dimension of $\mathfrak{F}(A,\nu)$ is the Brown measure of $\{z \in \mathbb{C} \mid |z| \geq \nu\}$ [20, Lemma 7.18].

For our purposes, we use these subspaces to isolate in the unit circle in the integral representation of $\log\det_G(I - A^k)$.

Theorem 4.4. *Let $A\colon U \to U$ be a morphism of Hilbert–G–modules and let μ_A denote the Brown measure of A. For $\nu > 1$, we set $\mathfrak{K}_\nu = \mathfrak{E}(A,\nu) \cap \mathfrak{F}(A,\nu^{-1})$. Then for $k \geq 1$ we have:*

$$\log \det_G\big(I - A^k\big)\big|_{\mathfrak{K}_\nu} = \int_{\nu^{-1} \leq |z| \leq \nu} \log\left|1 - z^k\right| \, d\mu_A.$$

Proof. Using the function $h(z) = 1 - z^k$, by Theorem 4.1 (3) we have:

$$\log \det_G\big(I - A^k\big)\big|_{\mathfrak{K}_\nu} = \int_{\mathbb{C}} \log\left|1 - z^k\right| \, d\mu_{A|_{\mathfrak{K}_\nu}}.$$

Let $P\colon U \to U$ denote the projection to the orthogonal complement of $\mathfrak{K}_\nu$. Then $\mu_A = \mu_{A|_{\mathfrak{K}_\nu}} + \mu_{PAP}$ (cf. [20, Remark 7.17]). Let $\mathbb{C}_\nu = \{z \in \mathbb{C} \mid \nu^{-1} \leq |z| \leq \nu\}$. According to [20, Main Theorem 1.1], we have $\operatorname{supp}(\mu_{A|_{\mathfrak{K}_\nu}}) \subseteq \mathbb{C}_\nu$ and $\operatorname{supp}(\mu_{PAP}) \subseteq \mathbb{C} - \mathbb{C}_\nu$. Therefore we have:

$$\begin{aligned}
\log \det_G\big(I - A^k\big)\big|_{\mathfrak{K}_\nu} &= \int_{\mathbb{C}} \log\left|1 - z^k\right| \, d\mu_{A|_{\mathfrak{K}_\nu}} \\
&= \int_{\nu^{-1} \leq |z| \leq \nu} \log\left|1 - z^k\right| \, d\mu_{A|_{\mathfrak{K}_\nu}} \\
&= \int_{\nu^{-1} \leq |z| \leq \nu} \log\left|1 - z^k\right| \, d\mu_A. \qquad \square
\end{aligned}$$

5. Dynamics on the quasi-fixed submodule

Using the setting and notation from Section 2.2, we define the following Hilbert–$\mathbb{F}$–modules:

$$C_1^{(2)}(\widetilde{\Gamma}, \widetilde{H}) = L^2(\mathbb{F}) \otimes_{\mathbb{Q}[\mathbb{F}]} C_1(\widetilde{\Gamma}, \widetilde{H}; \mathbb{Q}), \quad V_{\mathrm{h}}^{(2)} = L^2(\mathbb{F}) \otimes_{\mathbb{Q}[\mathbb{F}]} V_{\mathrm{h}}$$

$$\text{and } V_{\mathrm{qf}}^{(2)} = L^2(\mathbb{F}) \otimes_{\mathbb{Q}[\mathbb{F}]} V_{\mathrm{qf}}.$$

We have that $C_1^{(2)}(\widetilde{\Gamma}, \widetilde{H}) = V_{\mathrm{h}}^{(2)} + V_{\mathrm{qf}}^{(2)}$. The homomorphism $A_{f,H}\colon C_1(\widetilde{\Gamma}, \widetilde{H}; \mathbb{Q}) \to C_1(\widetilde{\Gamma}, \widetilde{H}, \mathbb{Q})$ extends to a bounded operator $A_{f,H}\colon C_1^{(2)}(\widetilde{\Gamma}, \widetilde{H}) \to C_1^{(2)}(\widetilde{\Gamma}, \widetilde{H})$. We will use a fixed isomorphism $C_1^{(2)}(\widetilde{\Gamma}, \widetilde{H}) \cong L^2(\mathbb{F})^{n_H}$ using a basis as in Section 3.3 and consider $V_{\mathrm{h}}^{(2)}$ and $V_{\mathrm{qf}}^{(2)}$ as submodules of $L^2(\mathbb{F})^{n_H}$.

In this section we explore the dynamics of $A_{f,H}$ on $V_{\mathrm{qf}}^{(2)}$. We remark that $A_{f,H}$ is not $\mathbb{F}$–equivariant. The main result is the following.

Theorem 5.1. *Suppose that the homotopy equivalence $f\colon \Gamma \to \Gamma$ satisfies the chain flare condition relative to the f–invariant graph $H \subset \Gamma$. Then there is a constant $C > 0$ such that for any $\xi \in V_{\mathrm{qf}}^{(2)}$ and $k \geq 0$ we have $C^{-1} \|\xi\| \leq \left\|A_{f,H}^k(\xi)\right\| \leq C \|\xi\|$.*

The only item of the chain flare condition that is needed for Theorem 5.1 is (CFH3). The estimate in Theorem 5.1 is used in the next section in the proof of Theorem 6.3 to show the equality between $V_{\mathrm{qf}}^{(2)}$ and the intersection $\mathfrak{E}(A_{f,H},\nu) \cap \mathfrak{F}(A_{f,H},\nu^{-1})$ for ν sufficient close to and greater than 1 when f satisfies the chain flare condition.

There are two cases to consider based on whether or not the Nielsen 1–chain generating V_{qf} is non-geometric or geometric (as the theorem obviously holds when $V_{\mathrm{qf}} = \{0\}$). These two cases are proved in Section 5.1 (Proposition 5.3) and Section 5.2 (Proposition 5.12) respectively. The key idea in both sections is to bound $\left\|A_{f,H}^k(x)\right\|$ for $x \in V_{\mathrm{qf}}$ and $k \geq 0$ in terms of the rational coefficients used to express x as a linear combination of translates of ρ independent of k.

5.1. **The Non-Geometric Case.** In this section, we assume $f\colon \Gamma \to \Gamma$ is a homotopy equivalence, $H \subset \Gamma$ is a f–invariant subgraph and $\rho = \pi_H^{\perp}([u,v]) \in C_1(\widetilde{\Gamma}, \widetilde{H}; \mathbb{Q})$ is a Nielsen 1–chain that is non-geometric. As previously stated, the idea is to bound the norm of $A^k_{f,H}(x)$ in terms of the rational coefficients expressing x as a linear combination of the translates of ρ independent of k. In this case, condition (NNC1) provides the existence of an edge that is in the support for only a single translate of ρ, which makes the calculation straightforward.

Lemma 5.2. *There is a constant $B \geq 1$ such that if $x = q_1 g_1 \rho + \cdots + q_r g_r \rho \in V_{\mathrm{qf}}$ and $k \geq 0$, then:*

$$\sum_{j=1}^{r} q_j^2 \leq \left\| A^k_{f,H}(x) \right\|^2 \leq B \sum_{j=1}^{r} q_j^2.$$

Proof. Let $e_\rho \in \mathsf{E}(\widetilde{\Gamma}) - \mathsf{E}(\widetilde{H})$ be an edge such that $\rho_{e_\rho} = \pm 1$ and $\rho_{g e_\rho} = 0$ for any non-trivial $g \in \mathbb{F}$. Such an edge exists by (NNC1). To simplify notation, we denote $A_{f,H}$ by A in the proof.

As $A(\rho) = \rho$, for $k \geq 0$ we have $A^k(x) = q_1 \Phi_f^k(g_1)\rho + \cdots + q_r \Phi_f^k(g_r)\rho$, and hence it suffices to prove the lemma for $k = 0$ as the required bounds depend only on the rational coefficients and not the group elements determining the translates.

On one hand, observe that $x_{g_j e_\rho} = \pm q_j$ since $(g_j \rho)_{g_j e_\rho} = \pm 1$ and $(g\rho)_{g_j e_\rho} = 0$ for all $g \in \mathbb{F}$ not equal to g_j. Thus:

$$\sum_{j=1}^{r} q_j^2 = \sum_{j=1}^{r} x^2_{g_j e_\rho} \leq \|x\|^2 .$$

On the other hand, letting $d = \#\,|\mathrm{supp}(\rho)|$ we observe that for any edge $e \in \mathsf{E}(\widetilde{\Gamma}) - \mathsf{E}(\widetilde{H})$, $(g\rho)_e = \rho_{g^{-1}e} \neq 0$ for at most d elements $g \in \mathbb{F}$. Thus we find:

$$x_e^2 = \left(\sum_{j=1}^{r} q_j (g_j \rho)_e \right)^2 \leq d^2 \max\{ q_j^2 \mid (g_j\rho)_e \neq 0 \}.$$

Additionally, each index $1 \leq j \leq r$ can realize the maximum value for at most d edges as well. Organizing the edges in $\mathrm{supp}(x)$ based on which index j provides the maximal value on the given edge, we find:

$$\|x\|^2 = \sum_{e \in \mathrm{supp}(x)} x_e^2 \leq d^3 \sum_{j=1}^{r} q_j^2.$$

Setting $B = d^3$ completes the proof. □

With this estimate, we can prove Theorem 5.1 when V_{qf} is generated by a non-geometric Nielsen 1–chain.

Proposition 5.3. *If V_{qf} is generated by a non-geometric Nielsen 1–chain, then there is a constant $C > 0$ such that for any $\xi \in V^{(2)}_{\mathrm{qf}}$ and $k \geq 0$ we have $C^{-1}\|\xi\| \leq \left\| A^k_{f,H}(\xi) \right\| \leq C\|\xi\|$.*

Proof. Let $B \geq 1$ be the constant from Lemma 5.2 and set $C = \sqrt{B}$. To simplify notation, we denote $A_{f,H}$ by A in the proof.

We first prove the proposition for $x \in V_{\mathrm{qf}}$. Write $x = q_1 g_1 \rho + \cdots + q_r g_r \rho$ for some rational numbers $q_1, \ldots, q_r \in \mathbb{Q}$ and elements $g_1, \ldots, g_r \in \mathbb{F}$. By Lemma 5.2 we find for any $k \geq 0$:

$$\left\|A^k(x)\right\|^2 \leq B \sum_{j=1}^{r} q_j^2 \leq B \|x\|^2, \text{ and}$$
$$\|x\|^2 \leq B \sum_{j=1}^{r} q_j^2 \leq B \left\|A^k(x)\right\|^2.$$

This proves the proposition for $x \in V_{\mathrm{qf}}$.

Now given $\xi \in V_{\mathrm{qf}}^{(2)}$, $k \geq 0$ and $\epsilon > 0$, there exists $x_1, x_2 \in V_{\mathrm{qf}}$ such that both of

$$\left|\|\xi\|^2 - \|x_1 + ix_2\|^2\right|, \text{ and } \left|\left\|A^k(\xi)\right\|^2 - \left\|A^k(x_1 + ix_2)\right\|^2\right|$$

are less than ϵ. As A is a real operator, we have $\left\|A^k(x_1 + ix_2)\right\|^2 = \left\|A^k(x_1)\right\|^2 + \left\|A^k(x_2)\right\|^2$. Therefore:

$$\begin{aligned}\left\|A^k(\xi)\right\|^2 &\leq \left\|A^k(x_1 + ix_2)\right\|^2 + \epsilon \\ &= \left\|A^k(x_1)\right\|^2 + \left\|A^k(x_2)\right\|^2 + \epsilon \\ &\leq B\|x_1\|^2 + B\|x_2\|^2 + \epsilon \\ &\leq B\|\xi\|^2 + \epsilon(B+1).\end{aligned}$$

Similarly:

$$\begin{aligned}\|\xi\|^2 &\leq \|x_1 + ix_2\|^2 + \epsilon \\ &\leq \|x_1\|^2 + \|x_2\|^2 + \epsilon \\ &\leq B\left\|A^k(x_1)\right\|^2 + B\left\|A^k(x_2)\right\|^2 + \epsilon \\ &\leq B\left\|A^k(\xi)\right\|^2 + \epsilon(B+1).\end{aligned}$$

As this holds for all $\epsilon > 0$, we have $C^{-1}\|\xi\| \leq \left\|A^k(\xi)\right\| \leq C\|\xi\|$ as desired. □

5.2. **The Geometric Case.** In this section, we assume $f\colon \Gamma \to \Gamma$ is a homotopy equivalence, $H \subset \Gamma$ is a f–invariant subgraph and $\rho = \pi_H^{\perp}([u,v]) \in C_1(\widetilde{\Gamma}, \widetilde{H}; \mathbb{Q})$ is a Nielsen 1–chain that is geometric and that generates V_{qf}. Again, as previously stated, the key idea is to bound the norm of $A_{f,H}^k(x)$ in terms of the rational coefficients expressing x as a linear combination of the translates of ρ independent of k. In this case, we will work with an auxiliary graph T_ρ that captures the combinatorics of the translates of ρ. The graph T_ρ has a free action by $\mathbb{F}$ and we consider the $\mathbb{Q}[\mathbb{F}]$–modules of compactly supported 0– and 1–cochains $C_c^0(T_\rho; \mathbb{Q})$ and $C_c^1(T_\rho; \mathbb{Q})$ respectively. Of importance is the coboundary operator $\delta_0\colon C_c^0(T_\rho; \mathbb{Q}) \to C_c^1(T_\rho; \mathbb{Q})$ which is a $\mathbb{Q}[\mathbb{F}]$–module homomorphism given by:

$$\delta_0\psi(\varepsilon) = \psi(\mathsf{t}(\varepsilon)) - \psi(\mathsf{o}(\varepsilon)).$$

where $\varepsilon \in \mathsf{E}(T_\rho)$. We will consider the usual L^2–norms on both $C_c^0(T_\rho; \mathbb{Q})$ and $C_c^1(T_\rho; \mathbb{Q})$. As our cochains are compactly support, these norms is well-defined.

We will define a "realization" map $R\colon V_{\mathrm{qf}} \to C_c^0(T_\rho; \mathbb{Q})$ and show that the sum of the squares of the rational coefficients of x equals $\|R(x)\|^2$ and also that $\|x\|$ equals $\|\delta_0 R(x)\|$. This takes place in

Lemma 5.10. There is bi-Lipschitz relation between $\|\psi\|$ and $\|\delta_0\psi\|$ that we recall in Lemma 5.11. This gives us the desired relation between the sum of the squares of the rational coefficients of x and $\|x\|^2$ from which Proposition 5.12 follows in a similar way to Proposition 5.3.

The graph T_ρ is defined by the following data.

$$\mathsf{V}(T_\rho) = \mathbb{F}$$
$$\mathsf{E}(T_\rho) = \{[g_1, g_2] \mid \mathrm{supp}(g_1\rho) \cap \mathrm{supp}(g_2\rho) \neq \emptyset\}$$

The graph T_ρ is not connected in general, but if T_0 and T_1 are components of T_ρ, then there is an element $g \in \mathbb{F}$ such that $T_1 = gT_0$. We note that there is a bijection between the edges of T_ρ and the edges $e \in \mathsf{E}(\widetilde{\Gamma}) - \mathsf{E}(\widetilde{H})$. Indeed, (GNC1) implies the assignment that sends an edge $[g_1, g_2] \in \mathsf{E}(T_\rho)$ to $\mathrm{supp}(g_1\rho) \cap \mathrm{supp}(g_2\rho)$ defines a function from $\mathsf{E}(T_\rho) \to \mathsf{E}(\widetilde{\Gamma}) - \mathsf{E}(\widetilde{H})$ and (GNC2) implies this function is a bijection.

Example 5.4. Let Γ be the theta graph labeled as in Figure 2. In this example, the homotopy equivalence $f\colon \Gamma \to \Gamma$ and subgraph $H \subset \Gamma$ are irrelevant and will not be specified.

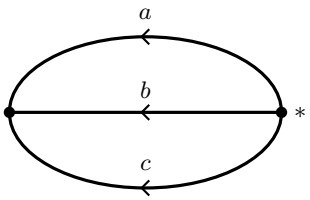

FIGURE 2. The graph Γ in Example 5.4.

There is an isomorphism $\pi_1(\Gamma, *) \cong \mathbb{F} = \langle x_1, x_2\rangle$ where x_1 corresponds to the edge-path $a\bar{b}$ and x_2 corresponds to th edge-path $c\bar{a}$. Fix a lift $\tilde{*}$ of $*$ to $\widetilde{\Gamma}$ that lies on the axes of x_1 and x_2. We consider the edge-path from $\tilde{*}$ to $x_1x_2x_1^{-1}x_2^{-1}\tilde{*}$. We fix lifts of a, b and c respectively in $\widetilde{\Gamma}$ as pictured in Figure 4 and abusing notation continue to denote them by a, b and c respectively. Then the 1–chain:

$$\rho = [\tilde{*}, x_1x_2x_1^{-1}x_2^{-1}\tilde{*}] = a - b + x_1c - x_1x_2a + x_1x_2x_1^{-1}b - x_1x_2x_1^{-1}x_2^{-1}c$$

satisfies conditions (GNC1), (GNC2) and (GNC3). Indeed, (GNC1) is apparent since the edge-path $a\bar{b}c\bar{a}b\bar{c}$ in Γ does not repeat a two-letter subword, (GNC2) is apparent as $\mathrm{supp}(\rho)$ contains exactly two edges from each orbit of edges in $\widetilde{\Gamma}$ and (GNC3) holds for $g_1 = x_1x_2$ and $g_2 = x_1x_2x_1^{-1}$. The six translates of ρ whose support has non-empty intersection with the support of ρ are also illustrated in Figure 3. The corresponding portion of T_ρ is illustrated in Figure 4.

There is an obvious map $R'\colon V_{\mathrm{qf}} \to C_c^0(T_\rho; \mathbb{Q})$ that sends a 1–chain $x = q_1g_1\rho + \cdots + q_rg_r\rho \in V_{\mathrm{qf}}$ to the function $R'(x) = q_1\chi_{g_1} + \cdots + q_r\chi_{g_r}$ where χ_g is the characteristic function of the set $\{g\} \subset \mathsf{V}(T_\rho)$. It is clearly true that $\sum_{j=1}^r q_j^2 = \|R'(x)\|^2$. Additionally, for the Nielsen 1–chain as in Example 5.4, we can demonstrate that $\|x\| = \|\delta_0 R'(x)\|$. Indeed, if e is the unique edge in $\mathrm{supp}(g\rho) \cap \mathrm{supp}(g'\rho)$, then, up to sign, the coefficient x_e is equal to $\sum_{j=1}^r q_j(\chi_{g'}(g_j) - \chi_g(g_j))$. (Notice that there are at most two nonzero terms in the sum.) This follows as for the Nielsen 1–chain in Example 5.4, for any edge, the coefficients of its translates in $\mathrm{supp}(\rho)$ are 1 and -1. Next, if $\varepsilon \in \mathsf{E}(T_\rho)$ is the edge corresponding to $\mathrm{supp}(g\rho) \cap \mathrm{supp}(g'\rho)$, then up to swapping the orientation of ε we have $\mathsf{o}(\varepsilon) = g$ and $\mathsf{t}(\varepsilon) = g'$, hence $\delta_0 R'(x)(\varepsilon)$ is equal to $\sum_{j=1}^r q_j(\chi_{g_j}(g') - \chi_{g_j}(g))$. (Again, there are at most

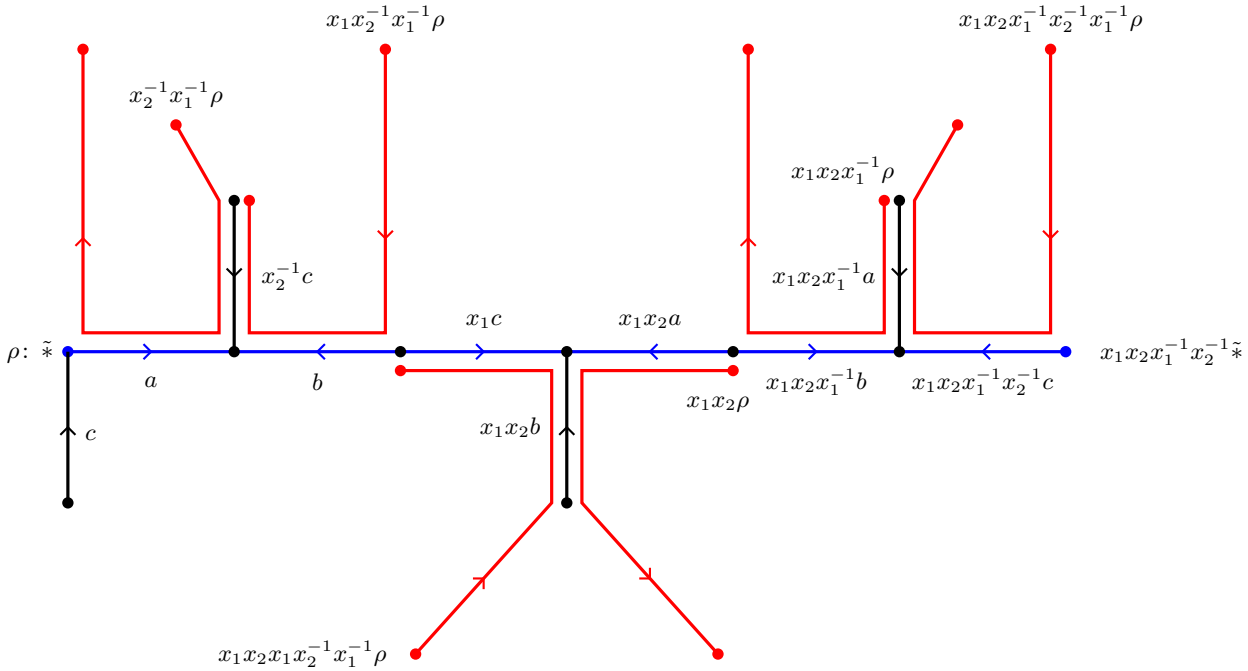

FIGURE 3. The Nielsen 1–chain ρ in Example 5.4 and the six translates whose support intersects the support of ρ.

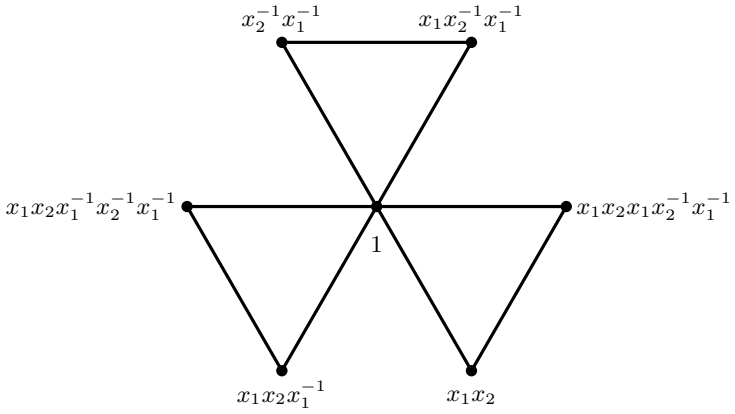

FIGURE 4. A portion of the graph T_ρ in Example 5.4.

two nonzero terms.) This shows that $\|\delta_0 R'(x)\| = \|x\|$ as claimed since $\chi_{g'}(g) = \chi_g(g')$ for any $g, g' \in \mathbb{F}$.

In general though, it is not the case that the coefficients of the translates of any edge in $\mathrm{supp}(\rho)$ are 1 and -1 and so we need to take such edges into account when defining the realization map $R\colon V_{\mathrm{qf}} \to C_c^0(T_\rho;\mathbb{Q})$. We call an edge $e \in \mathsf{E}(\widetilde{\Gamma}) - \mathsf{E}(\widetilde{H})$ *non-orientable* if the coefficients of its translates in $\mathrm{supp}(\rho)$ have the same sign, in which case they are either are both 1 or either both -1. Likewise, we call an edge $\varepsilon \in \mathsf{E}(T_\rho)$ non-orientable is the corresponding edge in $\mathsf{E}(\widetilde{\Gamma}) - \mathsf{E}(\widetilde{H})$ is non-orientable. In this language, there are no non-orientable edges in Example 5.4.

If the unique edge in $\mathrm{supp}(g_j\rho)\cap\mathrm{supp}(g_k\rho)$ is non-orientable, to get an equality $\|x\| = \|\delta_0 R(x)\|$, we need to make sure that we have $(R(x)(g_k) - R(x)(g_j))^2 = (q_k + q_j)^2$ and so one of the terms in the expression for $R(x)$ needs to be multiplied by -1. One way to accomplish this is the following. Fix a vertex $g_0 \in \mathsf{V}(T_\rho)$, let m denote the number of non-orientable edges in an edge-path from g_0 to the vertex $g_j \in \mathsf{V}(T_\rho)$ and use $(-1)^m q_j \chi_{g_j}$ in the summation formula for $R(x)$. If T_ρ is a tree, then this construction is obviously well-defined. However T_ρ is not a tree in general. Nonetheless, we can still show that the parity of the number of non-orientable edges in an edge path is well-defined.

To this end, suppose $\mathbf{p}\colon\ p_0, \dots, p_m$ is an edge-path in T_ρ where $p_0 = p_m$. Recall from Section 2.1 that notation this means the edge-path $\mathbf{p}$ visits the vertices $p_0, \dots, p_m$ in this order. This unambiguously defines an edge-path since T_ρ is a simplicial graph. To simplify the exposition, the index j in the remainder of this description is considered modulo m. Let $g_0, \dots, g_{m-1} \in \mathbb{F}$ be the elements such that $p_j = g_j$ for $0 \leq j < m$. For each $0 \leq j < m$, the intersection $\mathrm{supp}(g_{j-1}\rho) \cap \mathrm{supp}(g_j\rho)$ consists of a single edge $e_j \in \mathsf{E}(\widetilde{\Gamma}) - \mathsf{E}(\widetilde{H})$. Thus we have $e_j, e_{j+1} \in \mathrm{supp}(g_j\rho)$ for $0 \leq j < m$. An *orientation* on $\mathbf{p}$ is a choice of orientations on the edges e_j so that $(g_j\rho)_{e_j} = -(g_j\rho)_{e_{j+1}}$ for $0 \leq j < m$. Note that the orientations on the edges are not assumed to be $\mathbb{F}$–equivariant.

Lemma 5.5. *Suppose $\mathbf{p}\colon\ p_0, \dots, p_m$ is an edge-path in T_ρ where $p_0 = p_m$. Then either $p_j = p_k$ for some distinct indices $0 \leq j, k < m$ or there exists an orientation on $\mathbf{p}$.*

Proof. To simplify the exposition, subscript indices are considered modulo m in this proof.

There are elements $g_0, \dots, g_{m-1} \in \mathbb{F}$ such that $p_j = g_j$ for $0 \leq j < m$. For each $0 \leq j < m$, the intersection $\mathrm{supp}(g_{j-1}\rho) \cap \mathrm{supp}(g_j\rho)$ consists of a single edge $e_j \in \mathsf{E}(\widetilde{\Gamma}) - \mathsf{E}(\widetilde{H})$. We have $e_j, e_{j+1} \in \mathrm{supp}(g_j\rho)$ for $0 \leq j < m$. If $e_j = e_k$ for some distinct indices $0 \leq j, k < m$, then by (GNC2), the pair of elements $\{g_{j-1}, g_j\}$ equals the pair of elements $\{g_{k-1}, g_k\}$. If $k \equiv j+1 \pmod m$, then the lemma holds as either $p_k = p_j$ or $p_k = p_{j-1}$. Else, we see that the lemma holds as either $p_j = p_{k-1}$ or $p_j = p_k$.

Therefore, we can assume that the set of edges $e_0, \dots, e_m$ are distinct for the remainder of the proof.

For each $0 \leq j < m$, we have that the pair of edges e_j, e_{j+1} belong to the edge-path $g_j[u, v] \subset \widetilde{\Gamma}$. We locally orient e_j and e_{j+1} to point away from each other. In other words, there is a component of $\widetilde{\Gamma} - \{\mathsf{t}(e_j), \mathsf{t}(e_{j+1})\}$ that contains both $\mathsf{o}(e_j)$ and $\mathsf{o}(e_{j+1})$.

We claim that if these local choices are not consistent on $\mathbf{p}$ then $p_j = p_k$ for some distinct indices $0 \leq j, k < m$. Clearly, if these choices are consistent then $(g_j\rho)_{e_j} = -(g_j\rho)_{e_{j+1}}$ for all $0 \leq j < m$ and so they determine an orientation on $\mathbf{p}$.

If these local choices are not consistent, then there is some index $0 \leq j < m$ such that the local orientation on e_j induced from $g_{j-1}[u, v]$ does not equal the local orientation on e_j induced from $g_j[u, v]$. This implies that the edge e_j lies on the edge path from e_{j-1} to e_{j+1}. Indeed, let w be midpoint of the edge e_j, let w_{j-1} denote the initial vertex of e_j in the orientation induced from $g_{j-1}[u, v]$ and let w_j denote the initial vertex of e_j in the orientation induced from $g_j[u, v]$. Then the edge-path from e_{j-1} to w goes through w_{j-1} and the edge-path from e_{j+1} to w goes through w_j. As w_{j-1} and w_j are distinct, this shows that the edge-path from e_{j-1} to e_{j+1} contains w and hence also e_j as claimed.

Let $\mathbf{q}$ be the edge path in $\widetilde{\Gamma}$ from $g_{j-2}[u, v]$ to $g_{j+1}[u, v]$. This edge path contains e_j. This follows from (GNC1) as $e_{j-1} \subset g_{j-2}[u, v]$, $e_{j+1} \subset g_{j+1}[u, v]$ and e_j is contained in neither $g_{j-2}[u, v]$ nor $g_{j+1}[u, v]$. See Figure 5.

The union $\bigcup_{r=j+1}^{j-2} g_r[u, v]$ is connected as $g_r[u, v] \cap g_{r+1}[u, v]$ is nonempty for all $0 \leq r < m$. Hence, this union also contains the edge-path $\mathbf{q}$ and thus the edge e_j. This implies that $e_j \in$

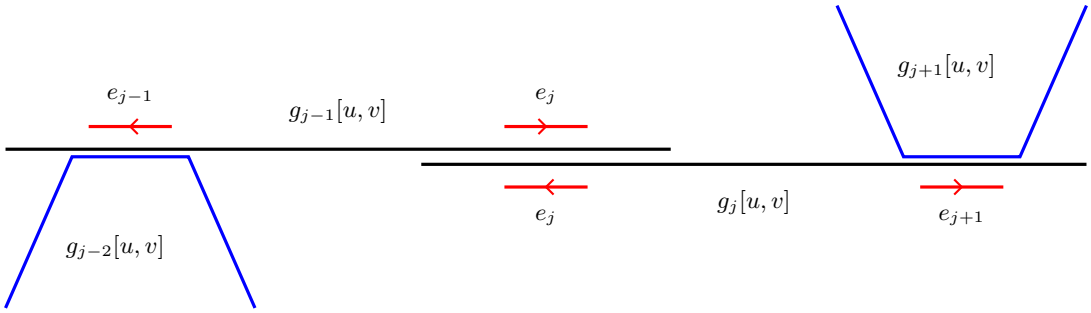

FIGURE 5. Inconsistent local orientations on the edge e_j in the proof of Lemma 5.5.

$\mathrm{supp}(g_k\rho)$ for some $k \neq j, j+1$. By (GNC2), we must have that $g_k = g_j$ or $g_k = g_{j+1}$. This shows that two vertices of $\mathbf{p}$ are the same and completes the proof. □

Motivated by the previous discussion, we introduce the following notion. Let $\mathbf{p}\colon p_0, \dots, p_m$ be an edge-path in T_ρ and let $g_0, \dots, g_m \in \mathbb{F}$ be the elements such that $p_j = g_j$ for $0 \leq j \leq m$. For each $1 \leq j \leq m$, the intersection $\mathrm{supp}(g_{j-1}\rho) \cap \mathrm{supp}(g_j\rho)$ consists of a single edge $e_j \in \mathsf{E}(\widetilde{\Gamma}) - \mathsf{E}(\widetilde{H})$. We have $(g_{j-1}\rho)_{e_j}, (g_j\rho)_{e_j} \in \{-1, 1\}$. We set $\sigma(\mathbf{p})$ to be equal to the number of indices $1 \leq j \leq m$ such that $(g_{j-1}\rho)_{e_j} = (g_j\rho)_{e_j}$. In other words, $\sigma(\mathbf{p})$ is the number of non-orientable edges along the edge-path $\mathbf{p}$. In particular, we have:

$$(-1)^{\sigma(\mathbf{p})} = \prod_{j=1}^{m} \left(-\frac{(g_{j-1}\rho)_{e_j}}{(g_j\rho)_{e_j}} \right). \tag{5.1}$$

We remark that $\sigma(\mathbf{p})$ is well-defined independent of choice of orientation on the edges e_j. By definition, if $\mathbf{p}$ is a trivial path, then $\sigma(\mathbf{p}) = 0$ so that $(-1)^{\sigma(\mathbf{p})} = 1$ and (5.1) holds where we define the empty product to be equal to 1.

We seek to show that $(-1)^{\sigma(\mathbf{p})}$ only depends on the endpoints of the edge-path $\mathbf{p}$. The next lemma shows this is true for orientable circuits. We extend this to all circuits in Lemma 5.7 and to all edge-paths in Corollary 5.8.

Lemma 5.6. *Suppose $\mathbf{p}\colon p_0, \dots, p_m$ is an edge-path in T_ρ where $p_0 = p_m$. If $\mathbf{p}$ is orientable, then $(-1)^{\sigma(\mathbf{p})} = 1$.*

Proof. To simplify the exposition, subscript indices are considered modulo m in this proof.

Let $g_0, \dots, g_{m-1} \in \mathbb{F}$ be the elements so that $p_j = g_j$ for $0 \leq j < m$. For each $0 \leq j < m$, there is an edge e_j that is the unique edge in $\mathrm{supp}(g_{j-1}\rho) \cap \mathrm{supp}(g_j\rho)$. As $\mathbf{p}$ is orientable, we may choose orientations on the edges e_j such that $(g_j\rho)_{e_j} = -(g_j\rho)_{e_{j+1}}$ for $0 \leq j < m$.

Thus, using (5.1), we find:

$$(-1)^{\sigma(\mathbf{p})} = \prod_{j=1}^{m} \left(-\frac{(g_{j-1}\rho)_{e_j}}{(g_j\rho)_{e_j}} \right) = \prod_{j=0}^{m-1} \left(-\frac{(g_j\rho)_{e_{j+1}}}{(g_j\rho)_{e_j}} \right) = 1. \qquad \square$$

Lemma 5.7. *Suppose that $\mathbf{p}\colon p_0, \dots, p_m$ is an edge-path in T_ρ where $p_0 = p_m$. Then $(-1)^{\sigma(\mathbf{p})} = 1$.*

Proof. Assume the statement of the lemma is false. Fix an edge-path $\mathbf{p}\colon p_0, \dots, p_m$ where $p_0 = p_m$ and $(-1)^{\sigma(\mathbf{p})} = -1$ with m minimal among all such edge-paths $\mathbf{p}$. To simplify the exposition, subscript indices are considered modulo m in this proof.

By Lemma 5.6, the path $\mathbf{p}$ is not orientable. Hence, by Lemma 5.5, there are distinct indices $0 \leq j, k < m$ such that $p_j = p_k$. We consider the following edge-paths: $\mathbf{p}'\colon p_j, p_{j+1}, \ldots, p_k$ and $\mathbf{p}''\colon p_k, p_{k+1}, \ldots, p_j$. Notice that $\sigma(\mathbf{p}) = \sigma(\mathbf{p}') + \sigma(\mathbf{p}'')$. Hence either $(-1)^{\sigma(\mathbf{p}')} = -1$ or $(-1)^{\sigma(\mathbf{p}'')} = -1$. However, this is a contradiction as $p_j = p_k$ and the lengths of $\mathbf{p}'$ and $\mathbf{p}''$ are less than m.

This contradiction proves the lemma. □

From this, it follows readily that $(-1)^{\sigma(\mathbf{p})}$ depends only on the endpoints of $\mathbf{p}$ as desired.

Corollary 5.8. *Suppose that* $\mathbf{p}\colon p_0, \ldots, p_m$ *and* $\mathbf{q}\colon q_0, \ldots, q_n$ *are edge-paths in* T_ρ *where* $p_0 = q_0$ *and* $p_m = q_n$. *Then* $(-1)^{\sigma(\mathbf{p})} = (-1)^{\sigma(\mathbf{q})}$.

Proof. Apply Lemma 5.7 to the edge-path $\mathbf{p}\overline{\mathbf{q}}$ and note that $\sigma(\mathbf{p}\overline{\mathbf{q}}) = \sigma(\mathbf{p}) + \sigma(\mathbf{q})$. □

In each component $T_0 \subset T_\rho$ we fix a vertex g_0. For $g \in \mathsf{V}(T_0)$, we fix some path $\mathbf{p}_g$ from g_0 to g and define $\operatorname{sgn}(g) = (-1)^{\sigma(\mathbf{p}_g)}$. By Corollary 5.8, the function $\operatorname{sgn}\colon \mathbb{F} \to \{-1, 1\}$ is well-defined. We can now define the realization map $R\colon V_{\mathrm{qf}} \to C_c^0(T_\rho; \mathbb{Q})$. Given $x = q_1 g_1 \rho + \cdots + q_r g_r \rho$, we set:

$$R(x) = \operatorname{sgn}(g_1) q_1 \chi_{g_1} + \cdots + \operatorname{sgn}(g_r) q_r \chi_{g_r}.$$

Example 5.9. Let Γ be the 3–rose labeled as in Figure 6. As in Example 5.4, the homotopy equivalence $f\colon \Gamma \to \Gamma$ and subgraph $H \subset \Gamma$ are irrelevant and will not be specified.

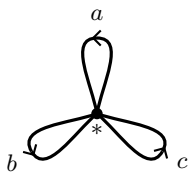

FIGURE 6. The graph Γ in Example 5.9.

There is an isomorphism $\pi_1(\Gamma, *) \cong \mathbb{F} = \langle x_1, x_2, x_3 \rangle$ where x_1 corresponds to the edge-path a and x_2 corresponds to the edge-path b and x_3 corresponds to the edge-path c. Fix a lift $\tilde{*}$ of $*$ to $\widetilde{\Gamma}$ that lies on the axes of x_1, x_2 and x_3. We consider the edge-path from $\tilde{*}$ to $x_1^2 x_2 x_3 x_2^{-1} x_3^{-1} \tilde{*}$. We fix lifts of a, b and c respectively in $\widetilde{\Gamma}$ as pictured in Figure 7 and abusing notation continue to denote them by a, b and c respectively. Then, in a similar way as in Example 5.4, we can see that the 1–chain:

$$\rho = [\tilde{*}, x_1^2 x_2 x_3 x_2^{-1} x_3^{-1} \tilde{*}] = a + x_1 a + x_1^2 b + x_1^2 x_2 c - x_1^2 x_2 x_3 x_2^{-1} b - x_1^2 x_2 x_3 x_2^{-1} x_3^{-1} c$$

satisfies conditions (GNC1), (GNC2) and (GNC3). Let $g_1 = x_1$, $g_2 = x_1^3 x_2 x_3 x_2^{-1} x_1^{-2}$ and $g_3 = x_1^2 x_2 x_3 x_2^{-1} x_1^{-2}$. Let $x = q_0\rho + q_1 g_1 \rho + q_2 g_2 \rho + q_3 g_3 \rho$. The 1–chain ρ along with these three translates are pictured in Figure 7. Using the vertex corresponding to the identity in T_ρ, we see that $\operatorname{sgn}(g_1) = -1$, $\operatorname{sgn}(g_2) = -1$ and $\operatorname{sgn}(g_3) = 1$ and thus we have:

$$R(x) = q_0 \chi_{\mathrm{id}_{\mathbb{F}}} - q_1 \chi_{g_1} - q_2 \chi_{g_2} + q_3 \chi_{g_3}.$$

For this ρ, the graph T_ρ is the 6–regular tree. The vertices and edges respectively with non-zero values for the 0–cochain $R(x)$ and its coboundary $\delta_0 R(x)$ respectively are shown in Figure 8. From this is it readily verified that $\|x\| = \|\delta_0 R(x)\|$.

We can now show that R has the properties mentioned previously in this section.

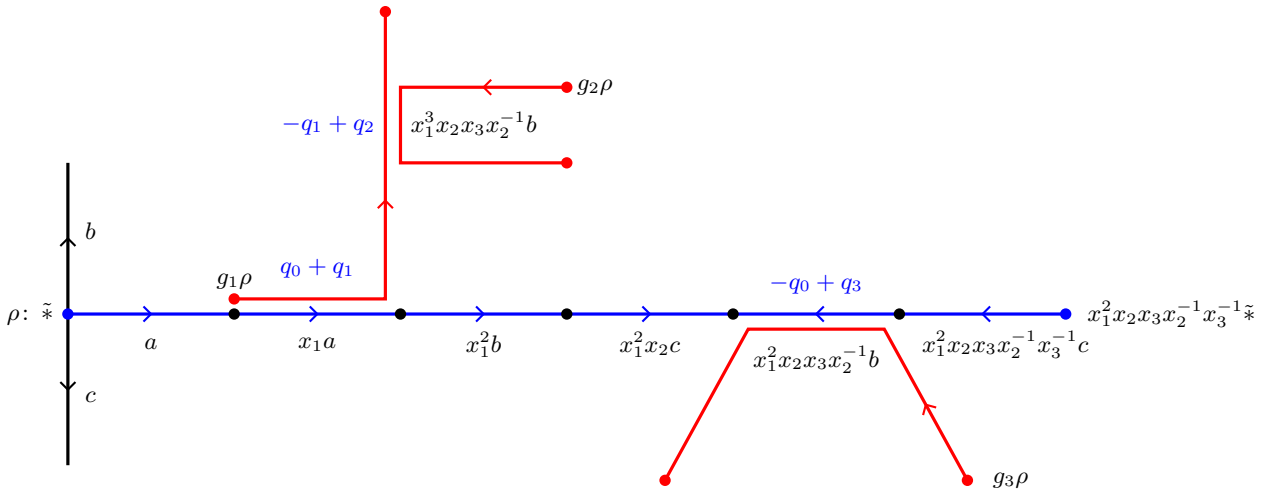

FIGURE 7. The Nielsen 1–chain in Example 5.9 and the translates determining the 1–chain x. The coefficient of x on the overlapping edges is also given.

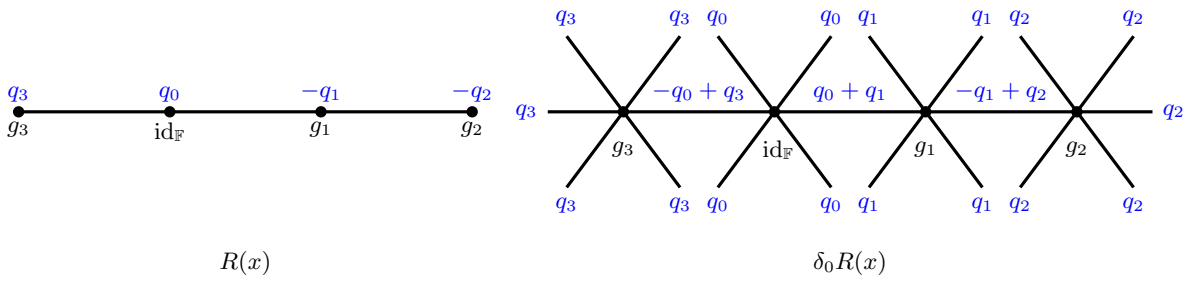

FIGURE 8. The 0–cochain $R(x)$ in Example 5.9 and its coboundary $\delta_0R(x)$. For $\delta_0R(x)$ the values are only defined up to sign as orientations on the edges are not specified.

Lemma 5.10. *The $\mathbb{Q}[\mathbb{F}]$–module homomorphism $R\colon V_{\mathrm{qf}} \to C_c^0(T_\rho;\mathbb{Q})$ satisfies the following statements for all $x \in V_{\mathrm{qf}}$.*

(1) $\|R(x)\| = \left\|R(A^k_{f,H}(x))\right\|$ *for all $k \geq 0$, and*
(2) $\|x\| = \|\delta_0 R(x)\|$.

Proof. To simplify notation, we denote $A_{f,H}$ by A in the proof.

We first verify (1). Let $x = q_1g_1\rho + \cdots + q_rg_r\rho \in V_{\mathrm{qf}}$. As $A(\rho) = \rho$, for $k \geq 0$ we have that $A^k(x) = q_1\Phi_f^k(g_1)\rho + \cdots + q_r\Phi_f^k(g_r)\rho$. Hence it is apparent that for all $k \geq 0$ that:

$$\left\|R(A^k(x))\right\|^2 = \sum_{j=1}^{r} q_j^2.$$

This shows (1).

Next, we verify (2). Let $x = q_1 g_1 \rho + \cdots + q_r g_r \rho \in V_{\mathrm{qf}}$. As $\mathrm{supp}(g\rho) \subset \mathsf{E}(\widetilde{\Gamma}) - \mathsf{E}(\widetilde{H})$ for all $g \in \mathbb{F}$, we have:

$$\|x\|^2 = \sum_{e \in \mathsf{E}(\widetilde{\Gamma})} x_e^2 = \sum_{e \in \mathsf{E}(\widetilde{\Gamma}) - \mathsf{E}(\widetilde{H})} \left(\sum_{j=1}^{r} q_j (g_j \rho)_e \right)^2 .$$

For the 1–cochain $\delta_0 R(x) \in C_c^1(T_\rho; \mathbb{Q})$, we have:

$$\|\delta_0 R(x)\|^2 = \sum_{\varepsilon \in \mathsf{E}(T_\rho)} \big(R(x)(\mathsf{t}(\varepsilon)) - R(x)(\mathsf{o}(\varepsilon)) \big)^2 .$$

As explained at the beginning of this section, an edge $e \in \mathsf{E}(\widetilde{\Gamma}) - \mathsf{E}(\widetilde{H})$ corresponds bijectively to the edge $\varepsilon \in \mathsf{E}(T_\rho)$ where $o(\varepsilon) = g$ and $t(\varepsilon) = g'$ and $g, g' \in \mathbb{F}$ are the unique elements so that e is the unique edge in the intersection $\mathrm{supp}(g\rho) \cap \mathrm{supp}(g'\rho)$. We will prove item (2) by showing that:

$$\left(\sum_{j=1}^{r} q_j (g_j \rho)_e \right)^2 = \big(R(x)(g') - R(x)(g) \big)^2 . \tag{5.2}$$

There are three cases depending on the cardinality of $\{g, g'\} \cap \{g_1, \ldots, g_r\}$.

If $\{g, g'\} \cap \{g_1, \ldots, g_r\} = \emptyset$, then $(g_j \rho)_e = 0$ for all $1 \leq j \leq r$ and $R(x)(g') = R(x)(g) = 0$ and thus both sides of (5.2) are equal to 0.

Next, suppose that $\{g, g'\} \cap \{g_1, \ldots, g_r\} = \{g_{j_1}\}$. Then $(g_j \rho)_e \neq 0$ only if $j = j_1$ and so the left-hand side of (5.2) is equal to $q_{j_1}^2$. Likewise, assuming without loss of generality that $g' = g_{j_1}$, we have $R(x)(g') = \mathrm{sgn}(g') q_{j_1}$ and $R(x)(g) = 0$ and so the righthand side of (5.2) is also equal to $q_{j_1}^2$.

Lastly, suppose that $\{g, g'\} \cap \{g_1, \ldots, g_r\} = \{g_{j_1}, g_{j_2}\}$. The left-hand side of (5.2) is equal to $(q_{j_1}(g_{j_1}\rho)_e + q_{j_2}(g_{j_2}\rho)_e)^2$. If $(g_{j_1}\rho)_e = (g_{j_2}\rho)_e$, i.e., e is non-orientable, then this quantity equals $(q_{j_1} + q_{j_2})^2$. Else if $(g_{j_1}\rho)_e = -(g_{j_2}\rho)_e$, then this quantity equals $(q_{j_1} - q_{j_2})^2$. Without loss of generality, we assume $g' = g_{j_1}$ and $g = g_{j_2}$ so that the righthand side of (5.2) equals $(\mathrm{sgn}(g_{j_1}) q_{j_1} - \mathrm{sgn}(g_{j_2}) q_{j_2})^2$. If $(g_{j_1}\rho)_e = (g_{j_2}\rho)_e$, then $\mathrm{sgn}(g_{j_2}) = -\mathrm{sgn}(g_{j_1})$ and so in this case, the righthand side equals $(q_{j_1} + q_{j_2})^2$. Else if $(g_{j_1}\rho)_e = -(g_{j_2}\rho)_e$, then $\mathrm{sgn}(g_{j_2}) = \mathrm{sgn}(g_{j_1})$ and so in this case the righthand side equals $(q_{j_1} - q_{j_2})^2$.

This completes the verification of (5.2) and thus completes the proof of item (2). □

We can now establish that the coboundary operator is bi-Lipschitz.

Lemma 5.11. *There is a constant $B \geq 1$ such that for any $\psi \in C_c^0(T_\rho; \mathbb{Q})$:*

$$B^{-1} \|\psi\| \leq \|\delta_0 \psi\| \leq B \|\psi\| .$$

Proof. Boundedness of the coboundary operator $\delta_0 \colon C_c^0(T_\rho; \mathbb{Q}) \to C_c^1(T_\rho; \mathbb{Q})$ is well-known and a simple calculation that we reproduce here. We note that any vertex in T_ρ has degree $d =$

$\#\,|\mathrm{supp}(\rho)|$. We compute:

$$\begin{aligned}\|\delta_0\psi\|^2 &= \sum_{\varepsilon\in\mathsf{E}(T_\rho)} \delta_0\psi(\varepsilon)^2 = \sum_{\varepsilon\in\mathsf{E}(T_\rho)} \big(\psi(\mathsf{t}(\varepsilon)) - \psi(\mathsf{o}(\varepsilon))\big)^2 \\ &\leq \sum_{\varepsilon\in\mathsf{E}(T_\rho)} 4\max\{\psi(\mathsf{t}(\varepsilon))^2, \psi(\mathsf{o}(\varepsilon))^2\} \\ &\leq 4d \sum_{g\in\mathsf{V}(T_\rho)} \psi(g)^2 = 4d\,\|\psi\|^2\,.\end{aligned}$$

The last inequality is observed by organizing edges based on which vertex provides the maximal value.

For the other direction, we fix a component $T_0 \subseteq T_\rho$ and let $\mathbb{F}_0 = \mathrm{stab}(T_0) \subseteq \mathbb{F}$. As any two components of T_ρ are isomorphic are graphs, it suffices to prove lower bound for $\psi \in C^0_c(T_0, \mathbb{Q})$. Without loss of generality, we assume that T_0 contains the vertex $\mathrm{id}_{\mathbb{F}}$. As $\mathbb{F}_0$ acts freely and transitively on the vertices of T_0, this implies that $\mathbb{F}_0$ is finitely generated. By (GNC3), $\mathbb{F}_0$ is nonabelian.

We consider the $\mathbb{Q}[\mathbb{F}_0]$–modules $C^0_c(T_0, \mathbb{Q})$ and $C^1_c(T_0; \mathbb{Q})$ respectively as submodules of the Hilbert spaces of square summable functions $\psi\colon \mathsf{V}(T_0) \to \mathbb{C}$, denoted $L^2(\mathsf{V}(T_0))$ and square summable functions $\psi\colon \mathsf{E}(T_0) \to \mathbb{C}$, denoted $L^2(\mathsf{E}(T_0))$, respectively. These are finite dimensional Hilbert–$\mathbb{F}_0$–modules isomorphic to $L^2(\mathbb{F}_0)$ and $L^2(\mathbb{F}_0)^n$ where n is the number of $\mathbb{F}_0$–orbits of edges in T_0. The calculation above shows that the coboundary operator extends to a bounded operator $\delta_0\colon L^2(\mathsf{V}(T_0)) \to L^2(\mathsf{E}(T_0))$, which is clearly $\mathbb{F}_0$–equivariant. It is easy to see that δ_0 is injective as the only functions with coboundary equal to the zero function are the constant functions and the only constant function which is square summable is the zero function. We will prove the lower bound for this operator. This will involve several definitions and some notation that is not need elsewhere and so we refer the reader to the book by Lück [25] for these details as cited below.

As $\mathbb{F}_0$ is nonabelian and acts free and cocompactly on T_0, the first Novikov–Shubin invariant $\alpha_1(T_0)$ is equal to $+\infty$ [25, Thoerem 2.55(5b)]. By [25, Lemma 2.3 & Lemma 2.4], this implies that there is a constant $\lambda > 0$ such that:

$$\dim_{\mathbb{F}_0}\left(\mathrm{im}\, E^{\partial_1\delta_0}_{\lambda^2}\right) = F_{\delta_0}(\lambda) = \dim_{\mathbb{F}_0}(\ker\delta_0) = 0.$$

(Here $\partial_1\colon L^2(\mathsf{E}(T_0))) \to L^2(\mathsf{V}(T_0))$ is the usual boundary operator—which is the adjoint of δ_0,$\{E^{\partial_1\delta_0}_\lambda\}$ is the spectral family of $\partial_1\delta_0$ and $F_{\delta_0}\colon [0,\infty) \to [0,\infty)$ is the spectral density function of δ_0.) Hence, $E^{\partial_1\delta_0}_{\lambda^2}(\psi) = 0$ for all $\psi \in L^2(\mathsf{V}(T_0))$ and by [25, Lemma 2.2(2)], this gives that $\|\delta_0\psi\| > \lambda\,\|\psi\|$ for all $\psi \in L^2(\mathsf{V}(T_0))$ which are non-zero. This is the desired lower bound.

Thus setting $B = \max\{2\sqrt{d}, \lambda^{-1}\}$ completes the proof of the lemma. □

With these estimates, the proof of Theorem 5.1 the geometric case is similar to the proof in the non-geometric case.

Proposition 5.12. *If V_{qf} is generated by a geometric Nielsen 1–chain, then there is a constant $C > 0$ such that for any $\xi \in V^{(2)}_{\mathrm{qf}}$ and $k \geq 0$ we have $C^{-1}\,\|\xi\| \leq \left\|A^k_{f,H}(\xi)\right\| \leq C\,\|\xi\|$.*

Proof. Let $B \geq 1$ be the constant from Lemma 5.11 and set $C = B^2$. To simplify notation, we denote $A_{f,H}$ by A in the proof.

We first prove the proposition for $x \in V_{\mathrm{qf}}$. By Lemmas 5.10 and 5.11 we find for any $k \geq 0$:

$$\left\|A^k(x)\right\| = \left\|\delta_0 R A^k(x)\right\| \leq B\left\|R(A^k(x))\right\| = B\,\|R(x)\| \leq B^2\,\|\delta_0 R(x)\| = C\,\|x\|\,.$$

Similarly for any $k \geq 0$:

$$\|x\| = \|\delta_0 R(x)\| \leq B \|R(x)\| = B \left\|R(A^k(x))\right\| \leq B^2 \left\|\delta_0 R(A^k(x))\right\| = C \left\|A^k(x)\right\|.$$

This proves the proposition for $x \in V_{\mathrm{qf}}$.

The general case $\xi \in V_{\mathrm{qf}}^{(2)}$ now proceeds exactly as in Proposition 5.3. □

6. Isolating the Quasi-Fixed Subspace

The purpose of this section is Theorem 6.3 which proves that $V_{\mathrm{qf}}^{(2)}$ equals the intersection $\mathfrak{E}(A_{f,H}, \nu) \cap \mathfrak{F}(A_{f,H}, \nu^{-1})$ for some ν sufficiently close to and greater than 1 when f satisfies the chain flare condition. We begin by showing that we can extend the chain flaring behavior from rational 1–chains to L^2–1–chains.

For $0 < \theta < 1$ we set:

$$N_\theta(V_{\mathrm{h}}^{(2)}) = \big\{\xi \in L^2(\mathbb{F})^{n_H} \mid \langle \Re(\xi), x\rangle > \theta \|\Re(\xi)\| \|x\| \text{ and } \langle \Im(\xi), x'\rangle > \theta \|\Im(\xi)\| \left\|x'\right\| \text{ for some } x, x' \in V_{\mathrm{h}}\big\}$$

where $\Re(\blacksquare)$ and $\Im(\blacksquare)$ denote the real part and imaginary part respectively.

Proposition 6.1. *Suppose that the homotopy equivalence $f\colon \Gamma \to \Gamma$ satisfies the chain flare condition relative to the f–invariant subgraph $H \subset \Gamma$. Then there exist constants $\lambda > 1$, $0 < \theta < 1$ and $N > 0$ such that for any $\xi \in N_\theta(V_{\mathrm{h}}^{(2)})^\infty$ we have:*

$$\lambda \left\|A_{f,H}^N(\xi)\right\| \leq \max\left\{\left\|A_{f,H}^{2N}(\xi)\right\|, \|\xi\|\right\}.$$

Proof. Let $\lambda_0 > 1$ and $0 < \theta_0 < 1$ be the constants from (CFH2) for rational 1–chains in $N_{\theta_0}(V_{\mathrm{h}})^\infty$. Let $N \in \mathbb{N}$ be such that $\lambda_0^N > 2$ and set $\lambda = \lambda_0^N/2$. Set $\theta = \frac{1+\theta_0}{2}$. To simplify notation, we denote $A_{f,H}$ by A in the proof.

By Lemma 2.2 (3), we have that for any rational 1–chain $x \in N_{\theta_0}(V_{\mathrm{h}})^\infty$:

$$2\lambda \left\|A^N(x)\right\| = \lambda_0^N \left\|A^N(x)\right\| \leq \max\left\{\left\|A^{2N}(x)\right\|, \|x\|\right\}.$$

Now fix a chain $\xi \in N_\theta(V_{\mathrm{h}}^{(2)})^\infty$ and decompose it as $\xi = \xi_1 + i\xi_2$ where $\xi_1 = \Re(\xi)$ and $\xi_2 = \Im(\xi)$. As A is a real operator, we have $\left\|A^N(\xi)\right\|^2 = \left\|A^N(\xi_1)\right\|^2 + \left\|A^N(\xi_2)\right\|^2$. Without loss of generality, we may assume that $\left\|A^N(\xi_1)\right\|^2 \geq \frac{1}{2}\left\|A^N(\xi)\right\|^2$.

Let $\epsilon > 0$. There is a rational 1–chain $x \in N_{\theta_0}(V_{\mathrm{h}})^\infty$ such that each of:

$$2\lambda \left\|A^N(\xi_1) - A^N(x)\right\|, \left\|A^{2N}(\xi_1) - A^{2N}(x)\right\|, \text{ and } \|\xi_1 - x\|$$

is less than ϵ. Using the observation above, we find:

$$\begin{aligned} 2\lambda \left\|A^N(\xi_1)\right\| &\leq 2\lambda \left\|A^N(x)\right\| + \epsilon \\ &\leq \max\left\{\left\|A^{2N}(x)\right\|, \|x\|\right\} + \epsilon \\ &\leq \max\left\{\left\|A^{2N}(\xi_1)\right\|, \|\xi_1\|\right\} + 2\epsilon. \end{aligned}$$

As this holds for all $\epsilon > 0$, we have $2\lambda \left\|A^N(\xi_1)\right\| \leq \max\left\{\left\|A^{2N}(\xi_1)\right\|, \|\xi_1\|\right\}$. Therefore:

$$\begin{aligned} \lambda \left\|A^N(\xi)\right\| &\leq 2\lambda \left\|A^N(\xi_1)\right\| \\ &\leq \max\left\{\left\|A^{2N}(\xi_1)\right\|, \|\xi_1\|\right\} \\ &\leq \max\left\{\left\|A^{2N}(\xi)\right\|, \|\xi\|\right\}. \end{aligned}$$

The last inequality again uses the fact that A is a real operator so that:

$$\left\|A^{2N}(\xi_1)\right\|^2 \leq \left\|A^{2N}(\xi_1)\right\|^2 + \left\|A^{2N}(\xi_2)\right\|^2 = \left\|A^{2N}(\xi)\right\|^2. \qquad \square$$

The proof of Lemma 2.2 carries over to L^2–1–chains $\xi \in N_\theta(V_{\mathrm{h}}^{(2)})^\infty$ using Proposition 6.1 in place of (CFH2).

Lemma 6.2. *Suppose that the homotopy equivalence $f\colon \Gamma \to \Gamma$ satisfies the chain flare condition relative to the f–invariant graph $H \subset \Gamma$ and let $\lambda > 1$, $0 < \theta < 1$ and $N > 0$ be the constants from Proposition 6.1. The following statements hold.*

(1) If $\xi \in N_\theta(V_{\mathrm{h}}^{(2)})^\infty$, $j \geq 1$ and $\lambda \left\|A_{f,H}^{jN}(\xi)\right\| \leq \left\|A_{f,H}^{(j-1)N}(\xi)\right\|$, then $\lambda^j \left\|A_{f,H}^{jN}(\xi)\right\| \leq \|\xi\|$.

(2) If $\xi \in N_\theta(V_{\mathrm{h}}^{(2)})^\infty$, $j \geq 1$ and $\lambda \|\xi\| \leq \left\|A_{f,H}^{N}(\xi)\right\|$, then $\lambda^j \|\xi\| \leq \left\|A_{f,H}^{jN}(\xi)\right\|$.

We can now prove the main result of this section.

Theorem 6.3. *Suppose that the homotopy equivalence $f\colon \Gamma \to \Gamma$ satisfies the chain flare condition relative to the f–invariant graph $H \subset \Gamma$. Then there is a constant $\lambda > 1$ such that for any $1 \leq \nu < \lambda$ we have*

$$V_{\mathrm{qf}}^{(2)} = \mathfrak{E}(A_{f,H}, \nu) \cap \mathfrak{F}(A_{f,H}, \nu^{-1}).$$

Proof. Let $\lambda_0 > 1$, $0 < \theta < 1$ and $N > 0$ be the constants from Proposition 6.1 and set $\lambda = \lambda_0^{1/N}$. Fix a number $1 \leq \nu < \lambda$. To simplify notation we denote $A_{f,H}$ by A in the proof.

We begin by showing that $V_{\mathrm{qf}}^{(2)} \subseteq \mathfrak{E}(A_{f,H}, \nu) \cap \mathfrak{F}(A_{f,H}, \nu^{-1})$. Suppose that $\xi \in V_{\mathrm{qf}}^{(2)}$. By Theorem 5.1, there is a $C > 0$ such that $C^{-1}\|\xi\| \leq \left\|A^j(\xi)\right\| \leq C\|\xi\|$ for all $j \geq 0$. Thus $\xi \in \mathfrak{E}(A, \nu)$ for any $\nu \geq 1$ as witnessed by the constant sequence $\xi_j = \xi$. Next, take a sequence $x_j \in \mathbb{C} \otimes V_{\mathrm{qf}}$ so that $\lim_{j\to\infty} \|x_j - \xi\| = 0$. For each j, there is a complex 1–chain $y_j \in \mathbb{C} \otimes V_{\mathrm{qf}}$ such that $A^j(y_j) = x_j$. Indeed, writing $x_j = z_{1,j} g_{1,j}\rho + \cdots + z_{r_j,j} g_{r_j,j}\rho$, we observe that $y_j = z_{1,j}\Phi_f^{-j}(g_{1,j})\rho + \cdots + z_{r_j,j}\Phi_f^{-j}(g_{r_j,j})\rho$ satisfies $A^j(y_j) = x_j$. Then $\lim_{j\to\infty} \left\|A^j(y_j) - \xi\right\| = 0$ and $\|y_j\| \leq C\left\|A^j(y_j)\right\| \leq 2C\|\xi\|$ for large enough j showing that $\xi \in \mathfrak{F}(A, \nu^{-1})$ for any $\nu \geq 1$ as witnessed by the sequence $\xi_j = y_j$. This shows that $V_{\mathrm{qf}}^{(2)} \subseteq \mathfrak{E}(A, \nu) \cap \mathfrak{F}(A, \nu^{-1})$.

We next demonstrate that $\mathfrak{E}(A_{f,H}, \nu) \cap \mathfrak{F}(A_{f,H}, \nu^{-1}) \cap V_{\mathrm{h}}^{(2)} = \{0\}$. From this it follows that $V_{\mathrm{qf}}^{(2)} = \mathfrak{E}(A, \nu) \cap \mathfrak{F}(A, \nu^{-1})$ as claimed since $L^2(\mathbb{F})^{n_H} = V_{\mathrm{h}}^{(2)} + V_{\mathrm{qf}}^{(2)}$.

To this end, we suppose that $\xi \in \mathfrak{F}(A, \nu^{-1}) \cap V_{\mathrm{h}}^{(2)}$ as witnessed by a sequence $(\xi_j) \subset L^2(\mathbb{F})^{n_H}$. In other words, $\lim_{j\to\infty} \left\|A^j(\xi_j) - \xi\right\| = 0$ and $\limsup_{j\to\infty} \|\xi_j\|^{1/j} \leq \nu$. As $\xi \in V_{\mathrm{h}}^{(2)}$, there is a $J \geq 0$ such that $A^j(\xi_j) \in N_\theta(V_{\mathrm{h}}^{(2)})$ for $j \geq J$. Hence $\xi_j \in N_\theta(V_{\mathrm{h}}^{(2)})^\infty$ for $j \geq J$. Let $S \subseteq \mathbb{N}$ be the subset of $j \geq J/N$ where:

$$\lambda_0 \left\|A^{jN}(\xi_{jN})\right\| \leq \left\|A^{(j-1)N}(\xi_{jN})\right\|.$$

By Lemma 6.2 (1), for $j \in S$ we have $\lambda_0^j \left\|A^{jN}(\xi_{jN})\right\| \leq \|\xi_{jN}\|$. If S is an infinite set, then

$$\limsup_{j\to\infty} \|\xi_{jN}\|^{1/jN} \geq \lambda_0^{1/N} = \lambda > \nu,$$

which contradicts the choice of the sequence (ξ_j). Hence for large enough j, by Proposition 6.1 we must have:

$$\lambda_0 \left\|A^{jN}(\xi_{jN})\right\| \leq \left\|A_f^{(j+1)N}(\xi_{jN})\right\|.$$

Taking the limit as $j \to \infty$ we find that $\lambda_0 \|\xi\| \leq \left\|A^N(\xi)\right\|$.

Next, we suppose that $\xi \in \mathfrak{E}(A,\nu) \cap V_{\mathrm{h}}^{(2)}$ as witnessed by a sequence $(\xi_j) \subset L^2(\mathbb{F})^{n_H}$. In other words, $\lim_{j\to\infty} \|\xi_j - \xi\| = 0$ and $\limsup_{j\to\infty} \left\|A^j(\xi_j)\right\|^{1/j} \leq \nu$. As A^N is a bounded operator, we also have that $\lim_{j\to\infty} \left\|A^N(\xi_j) - A^N(\xi)\right\| = 0$ and $\limsup_{j\to\infty} \left\|A^{N+j}(\xi_j)\right\|^{1/j} \leq \nu$. As $\xi \in V_{\mathrm{h}}^{(2)}$, there is a $J \geq 0$ such that $\xi_j \in N_\theta(V_{\mathrm{h}}^{(2)})$ for $j \geq J$. Hence $A^N(\xi_j) \in N_\theta(V_{\mathrm{h}}^{(2)})^\infty$ for $j \geq J$. Let $S \subseteq \mathbb{N}$ be the subset of $j \geq J/N$ where:

$$\lambda_0 \left\|A^N(\xi_{jN})\right\| \leq \left\|A^{2N}(\xi_{jN})\right\|.$$

By Lemma 6.2 (2), for $j \in S$, we have $\lambda_0^j \left\|A^N(\xi_{jN})\right\| \leq \left\|A^{(j+1)N}(\xi_{jN})\right\|$. If S is an infinite set, then

$$\limsup_{j\to\infty} \left\|A^{N+jN}(\xi_{jN})\right\|^{1/jN} \geq \lambda_0^{1/N} = \lambda > \nu,$$

which contradicts the choice of the sequence (ξ_j). Hence for large enough j, by Proposition 6.1 we must have:

$$\lambda_0 \left\|A^N(\xi_{jN})\right\| \leq \|\xi_{jN}\|.$$

Taking the limit as $j \to \infty$ we find that $\lambda_0 \left\|A^N(\xi)\right\| \leq \|\xi\|$.

Now consider $\xi \in \mathfrak{E}(A,\nu) \cap \mathfrak{F}(A,\nu^{-1}) \cap V_{\mathrm{h}}^{(2)}$. As $\xi \in \mathfrak{F}(A,\nu^{-1}) \cap V_{\mathrm{h}}^{(2)}$ we have $\lambda_0 \|\xi\| \leq \left\|A^N(\xi)\right\|$. As $\xi \in \mathfrak{E}(A,\nu) \cap V_{\mathrm{h}}^{(2)}$ we have $\lambda_0 \left\|A^N(\xi)\right\| \leq \|\xi\|$. Hence $\lambda_0^2 \|\xi\| \leq \|\xi\|$, which is impossible if $\xi \neq 0$ as $\lambda_0 > 1$. □

Remark 6.4. If $C_1(\widetilde{\Gamma}, \widetilde{H}; \mathbb{Q}) = V_{\mathrm{h}} \oplus V_{\mathrm{qf}}$ and the $\mathbb{Q}[\mathbb{F}]$–module V_{h} is $A_{f,H}$–invariant, then the conclusion of Theorem 6.3 holds under the weaker hypothesis of (CFH2) where we only insist that $x \in V_{\mathrm{h}}$. We sketch the modifications necessary for the proof of Theorem 6.3. Clearly, we still have $V_{\mathrm{qf}}^{(2)} \subseteq \mathfrak{E}(A_{f,H},\nu) \cap \mathfrak{F}(A_{f,H},\nu^{-1})$. Take $\xi \in \mathfrak{F}(A_{f,H},\nu) \cap V_{\mathrm{h}}^{(2)}$ as witnessed by a sequence $(\xi_j) \subset L^2(\mathbb{F})^{n_H}$. Write $\xi_j = \xi_j^{\mathrm{h}} + \xi_j^{\mathrm{qf}}$ where $\xi_j^{\mathrm{h}} \in V_{\mathrm{h}}^{(2)}$ and $\xi_j^{\mathrm{qf}} \in V_{\mathrm{qf}}^{(2)}$. As $A_{f,H}^j(\xi_j^{\mathrm{h}}) \in V_{\mathrm{h}}^{(2)}$ and $A_{f,H}^j(\xi_j^{\mathrm{qf}}) \in V_{\mathrm{qf}}^{(2)}$, we have $A_{f,H}^j(\xi_j^{\mathrm{h}}) \to \xi$ and $A_{f,H}^j(\xi_j^{\mathrm{qf}}) \to 0$. Since $\left\|\xi_j^{\mathrm{qf}}\right\| \leq C \left\|A_{f,H}^j(\xi_j^{\mathrm{qf}})\right\|$, we have $\xi_j^{\mathrm{qf}} \to 0$ and hence:

$$\limsup_{j\to\infty} \left\|A_{f,H}^j(\xi_j^{\mathrm{h}})\right\|^{1/j} = \limsup_{j\to\infty} \left\|A_{f,H}^j(\xi_j)\right\|^{1/j}.$$

In other words, we can assume that the sequence witnessing $\xi \in \mathfrak{F}(A_{f,H},\nu) \cap V_{\mathrm{h}}^{(2)}$ lies in $V_{\mathrm{h}}^{(2)}$. Thus as long as (CFH2) holds for elements of $V_{\mathrm{h}}^{(2)}$ we can still conclude that $\lambda_0 \|\xi\| \leq \left\|A^N(\xi)\right\|$.

A similar statement is true for $\xi \in \mathfrak{E}(A_{f,H},\nu) \cap V_{\mathrm{h}}^{(2)}$. Hence we may weaken the hypothesis on (CFH2) to $x \in V_{\mathrm{h}}$. The chain flare assumption is written with the subset $N_\theta(V_{\mathrm{h}})^\infty$ as it is not obvious how to find an invariant direct sum complement to V_{qf}.

7. The Restriction to the Quasi-Fixed Subspace

The purpose of this section is two-fold. Firstly, we will compute the determinant of the operator $I - L_{f,H}^k$ restricted to the subspace $W_{\mathrm{qf}}^{(2)} \subset L^2(G_\phi)^{n_H}$ corresponding to the quasi-fixed subspace $V_{\mathrm{qf}}^{(2)} \subset L^2(\mathbb{F})^{n_H}$. This takes place in Section 7.2 (Theorem 7.3). To make the statement precise, we first recall the notion of induction of Hilbert–G–modules and morphisms which takes place in Section 7.1. Secondly, we extend Theorem 6.3 to the operator $L_{f,H}$ showing the equality between $W_{\mathrm{qf}}^{(2)}$ and the intersection $\mathfrak{E}(L_{f,H},\nu) \cap \mathfrak{F}(L_{f,H},\nu^{-1})$ for ν sufficiently close to and greater than 1 when f satisfies the chain flare condition.

7.1. **Induction.** Let G be a countable group and H a subgroup of G. By $\iota\colon H \to G$ we denote the natural inclusion. Given a Hilbert–H–module U, the Hilbert space completion of $\mathbb{C}[G] \otimes_{\mathbb{C}[H]} U$ is a Hilbert–G–module denoted $\iota_* U$. A morphism $A\colon U \to V$ of Hilbert–H–modules induces a morphism $\iota_* A\colon \iota_* U \to \iota_* V$ in the obvious way. For more details see [25, Section 1.1.5]. The main properties of induction we use with regards to the Fuglede–Kadison determinant and the Brown measure are recorded below.

Lemma 7.1. *Let $\iota\colon H \to G$ be an injective group homomorphism and let $A\colon U \to V$ be a morphism of finite dimensional Hilbert–H–modules. The following statements hold.*

(1) $\det_G(\iota_* A) = \det_H(A)$.
(2) $\mu_{\iota_* A} = \mu_A$.

Proof. Item (1) is [25, Theorem 3.14 (6)].

Item (2) follows from (1) as μ_A is the Riesz measure associated to $\frac{1}{2\pi}\nabla^2 \log \det_H(A - zI)$ and $\mu_{\iota_* A}$ is the Riesz measure associated to $\frac{1}{2\pi}\nabla^2 \log \det_G(\iota_* A - zI)$ [8]. By (1) these two functions are identical. □

7.2. **The induced quasi-fixed subspace.** Let $f\colon \Gamma \to \Gamma$ be a homotopy equivalence and let $H \subset \Gamma$ be an f–invariant subgraph. We will use the set-up and notation from Section 3.3. In particular, we have an presentation of G_ϕ as a semi-direct product $G_\phi \cong \mathbb{F} \rtimes_{\Phi_f} \langle t \rangle$. The inclusion $\iota\colon \mathbb{F} \to G_\phi$ gives rise to an inclusion $V_{\mathrm{qf}}^{(2)} \subseteq L^2(\mathbb{F})^{n_H} \subseteq L^2(G_\phi)^{n_H}$ and we define $W_{\mathrm{qf}}^{(2)}$ to be the closure of $\bigoplus_{\ell \in \mathbb{Z}} t^\ell V_{\mathrm{qf}}^{(2)}$ in $L^2(G_\phi)^{n_H}$. In other words, $W_{\mathrm{qf}}^{(2)} = \iota_* V_{\mathrm{qf}}^{(2)}$.

Proposition 7.2. *If $W_{\mathrm{qf}}^{(2)}$ is nontrivial, then it is isomorphic to $L^2(G_\phi)$ as a Hilbert–G_ϕ–module.*

Proof. We will show that if $V_{\mathrm{qf}}^{(2)}$ is nontrivial then it is isomorphic to $L^2(\mathbb{F})$. This implies the statement of the proposition as $L^2(G_\phi) = \iota_* L^2(\mathbb{F})$.

Let $\rho = \pi_H^\perp([u, v]) \in C_1(\widetilde{\Gamma}, \widetilde{H}; \mathbb{Q})$ be the Nielsen 1–chain generating V_{qf}. We define an $\mathbb{F}$–equivariant operator $O\colon \mathbb{Q}[\mathbb{F}] \to V_{\mathrm{qf}}$. Given $x = q_1 g_1 + \cdots + q_r g_r \in \mathbb{Q}[\mathbb{F}]$, we set $O(x) = q_1 g_1 \rho + \cdots + q_r g_r \rho \in V_{\mathrm{qf}}$. Clearly, this map is a $\mathbb{Q}[\mathbb{F}]$–module surjective homomorphism. We will show that there is a constant $D \geq 1$ such that:

$$D^{-1} \|x\| \leq \|O(x)\| \leq D \|x\|$$

This shows that O is injective and extends to a Hilbert–$\mathbb{F}$–module isomorphism $O\colon L^2(\mathbb{F}) \to V_{\mathrm{qf}}^{(2)}$ as claimed. To this end, there are two cases depending on whether ρ is non-geometric or geometric.

Case 1: ρ is non-geometric. Let $B \geq 1$ be the constant from Lemma 5.2 and set $D = \sqrt{B}$. Given $x = q_1 g_1 + \cdots + q_r g_r \in \mathbb{Q}[\mathbb{F}]$ we have $\|x\|^2 = \sum_{j=1}^r q_j^2$. Applying the estimate in Lemma 5.2 with $k = 0$ we find:

$$\|x\| \leq \|O(x)\| \leq D \|x\|.$$

This proves the proposition in the the non-geometric case.

Case 2: ρ is geometric. We will make use of the realization map $R\colon V_{\mathrm{qf}} \to C_c^0(T_\rho; \mathbb{Q})$ and the sign map $\mathrm{sgn}\colon \mathbb{F} \to \{-1, 1\}$. Let $U\colon \mathbb{Q}[\mathbb{F}] \to C_c^0(T_\rho; \mathbb{Q})$ be the $\mathbb{Q}[\mathbb{F}]$–module isomorphism defined by:

$$U(q_1 g_1 + \cdots + q_r g_r) = q_1 \operatorname{sgn}(g_1) \chi_{g_1} + \cdots + q_r \operatorname{sgn}(g_r) \chi_{g_r}.$$

(Recall χ_g is the characteristic function of the set $\{g\} \subset \mathsf{V}(T_\rho)$.) We observe that $\|U(x)\| = \|x\|$ and that $U = RO$. Let C be the constant from Proposition 5.12 and set $D = \sqrt{C}$. As shown

in the proof of Proposition 5.12 with $k = 0$, we have $\|O(x)\| \leq D\,\|RO(x)\| \leq D^2\,\|O(x)\|$. As $\|x\| = \|U(x)\| = \|RO(x)\|$, this proves the proposition in this case. □

We recall the operator $L_{f,H}\colon L^2(G_\phi)^{n_H} \to L^2(G_\phi)^{n_H}$ defined in Section 3.3. This operator is easily expressed using the current set-up as follows. Given $\xi \in L^2(G_\phi)^{n_H}$, we write $\xi = \sum_{\ell\in\mathbb{Z}} t^\ell \xi^{(\ell)}$ where $\xi^{(\ell)} \in L^2(\mathbb{F})^{n_H}$. Then:

$$L_{f,H}(\xi) = \sum_{\ell\in\mathbb{Z}} t^{\ell+1} A_{f,H}(\xi^{(\ell)}). \tag{7.1}$$

In particular we see that $W^{(2)}_{\mathrm{qf}}$ is $L_{f,H}$–invariant as $V^{(2)}_{\mathrm{qf}}$ is $A_{f,H}$–invariant.

Theorem 7.3. *Suppose that the homotopy equivalence $f\colon \Gamma \to \Gamma$ satisfies the chain flare condition relative to the f–invariant graph $H \subset \Gamma$. Then for any $k \geq 0$, we have*

$$\log\det{}_{G_\phi}\big(I - L^k_{f,H}\big)\big|_{W^{(2)}_{\mathrm{qf}}} = 0.$$

Proof. If $W^{(2)}_{\mathrm{qf}} = \{0\}$ then the proposition holds as $\log\det_{G_\phi}(0) = 0$ by definition. Thus we assume that $W^{(2)}_{\mathrm{qf}} \neq \{0\}$ and we let $O\colon L^2(G_\phi) \to W^{(2)}_{\mathrm{qf}}$ be the isomorphism from Proposition 7.2.

Let $P\colon L^2(\langle t\rangle) \to L^2(\langle t\rangle)$ be the morphism given by right multiplication by $1 - t^k$. We have $\log\det_{\langle t\rangle}(P) = 0$ [25, Example 3.22]. Let $\iota'\colon \langle t\rangle \to G_\phi$ be the natural inclusion. We observe that $I - L^k_{f,H}\big|_{W^{(2)}_{\mathrm{qf}}} = O(\iota'_* P)O^{-1}$. Indeed, consider $x = t^\ell(q_1 g_1 \rho + \cdots + q_r g_r \rho) \in t^\ell V_{\mathrm{qf}}$. Then using the relation $t^k \Phi^k_f(g) = g t^k$ for all $g \in \mathbb{F}$ and $k \geq 0$ we find:

$$\begin{aligned}
(I - L^k_{f,H})(x) &= x - t^{\ell+k} A^k_{f,H}(q_1 g_1 \rho + \cdots + q_r g_r \rho) \\
&= x - t^{\ell+k}(q_1 \Phi^k_f(g_1)\rho + \cdots + q_r \Phi^k_f(g_r)\rho) \\
&= x - t^\ell(q_1 g_1 \rho + \cdots + q_r g_r \rho) t^k \\
&= x(1 - t^k).
\end{aligned}$$

Hence by Lemma 3.1 and Lemma 7.1 (1) we find:

$$\log\det{}_{G_\phi}\big(I - L^k_{f,H}\big)\big|_{W^{(2)}_{\mathrm{qf}}} = \log\det{}_{G_\phi}(\iota'_* P) = \log\det{}_{\langle t\rangle}(P) = 0.$$

as claimed. □

7.3. **Isolating the induced quasi-fixed subspace.** Using the expression in (7.1) for $L_{f,H}$ we can extend Theorem 6.3 to the operator $L_{f,H}$.

Theorem 7.4. *Suppose that the homotopy equivalence $f\colon \Gamma \to \Gamma$ satisfies the chain flare condition relative to the f–invariant graph $H \subset \Gamma$. Then there is a constant $\lambda > 1$ such that for any $1 \leq \nu < \lambda$ we have*

$$W^{(2)}_{\mathrm{qf}} = \mathfrak{E}(L_{f,H}, \nu) \cap \mathfrak{F}(L_{f,H}, \nu^{-1}).$$

Proof. Let λ be the constant from Theorem 6.3. Fix a number $1 \leq \nu < \lambda$. To simplify notation we denote $L_{f,H}$ by L and $A_{f,H}$ by A in the proof.

The proof that $W^{(2)}_{\mathrm{qf}} \subseteq \mathfrak{E}(L, \nu) \cap \mathfrak{F}(L, \nu^{-1})$ is similar to the proof of $V^{(2)}_{\mathrm{qf}} \subseteq \mathfrak{E}(A, \nu^{-1}) \cap \mathfrak{F}(A, \nu)$ in Theorem 6.3. Indeed, by (7.1), it is apparent that $C^{-1}\,\|\xi\| \leq \big\|L^j(\xi)\big\| \leq C\,\|\xi\|$ for the same constant $C \geq 1$ and for all $j \geq 0$.

Now suppose that $\xi \in \mathfrak{F}(L, \nu^{-1})$ as witnessed by the sequence $(\xi_j) \subset L^2(G_\phi)^{n_H}$. In other words, $\lim_{j\to\infty} \big\|L^j(\xi_j) - \xi\big\| = 0$ and $\lim_{j\to\infty} \|\xi_j\|^{1/j} \leq \nu$. Using the decompositions $\xi = \sum_{\ell\in\mathbb{Z}} t^\ell \xi^{(\ell)}$ and

$\xi_j = \sum_{\ell\in\mathbb{Z}} t^\ell \xi_j^{(\ell)}$ for each j, we have that $L^j(\xi_j) = \sum_{\ell\in\mathbb{Z}} t^{\ell+j} A^j(\xi_j^{(\ell)})$. Hence $\lim_{j\to\infty} \left\| A^j(\xi_j^{(\ell)}) - \xi^\ell \right\| = 0$ in $L^2(\mathbb{F})^{n_H}$ and

$$\limsup_{j\to\infty} \left\|\xi_j^{(\ell)}\right\|^{1/j} \leq \limsup_{j\to\infty} \|\xi_j\|^{1/j} \leq \nu$$

so that $\xi^{(\ell)} \in \mathfrak{F}(A, \nu^{-1})$ for each ℓ.

Similarly, if $\xi \in \mathfrak{E}(L, \nu)$ then writing $\xi = \sum_{\ell\in\mathbb{Z}} t^\ell \xi^{(\ell)}$ we have that $\xi^{(\ell)} \in \mathfrak{E}(A, \nu)$ as well for each $\ell \in \mathbb{Z}$.

Hence if $\xi \in \mathfrak{E}(L, \nu) \cap \mathfrak{F}(L, \nu^{-1})$ then writing $\xi = \sum_{\ell\in\mathbb{Z}} t^\ell \xi^{(\ell)}$, for each $\ell \in \mathbb{Z}$ we have $\xi^{(\ell)} \in V^{(2)}_{\mathrm{qf}}$ by Theorem 6.3 and so $\xi \in W^{(2)}_{\mathrm{qf}}$ as desired. □

8. Proof of Theorem 1.1

In this section we prove Theorem 1.1, the main theorem of the article. It is restated below for convenience.

Theorem 1.1. *Suppose that $f\colon \Gamma \to \Gamma$ is a homotopy equivalence that respects the reduced filtration $\emptyset = \Gamma_0 \subset \Gamma_1 \subset \cdots \subset \Gamma_S = \Gamma$ and that $f\colon \Gamma \to \Gamma$ represents the outer automorphism $\phi \in \mathrm{Out}(\mathbb{F})$. If the restriction of f to Γ'_s satisfies the chain flare condition relative to $\Gamma'_s \cap \Gamma_{s-1}$ for each $s \in \mathcal{EG}(f)$, then*

$$-\rho^{(2)}(G_\phi) = \sum_{s\in\mathcal{EG}(f)} \int_{1<|z|} \log|z|\, d\mu_{L_{f,s}}. \tag{8.1}$$

Moreover, each integral in (8.1) is positive and hence $-\rho^{(2)}(G_\phi) > 0$.

Proof. Let $f\colon \Gamma \to \Gamma$ be as in the statement of the theorem. By Theorem 3.2, for all $k \geq 1$ we have $\rho^{(2)}(G_{\phi^k}) = k\rho^{(2)}(G_\phi)$. According to [9, Remark 4.8], for all $k \geq 1$, we have $t^k J_1(f^k)_s = (tJ_1(f)_s)^k$. This implies that $L^k_{f,s} = \iota_* L_{f^k,s}$ where $\iota\colon G_{\phi^k} \to G_\phi$ is the natural inclusion. According to [8, Theorem 4.1] $\mu_{L^k_{f,s}}$ is the push-forward of $\mu_{L_{f,s}}$ under the map $z \mapsto z^k$. Hence using Lemma 7.1 (2) we have:

$$\int_{|z|>1} \log|z|\ d\mu_{L_{f^k,s}} = \int_{|z|>1} \log|z|\ d\mu_{L^k_{f,s}} = \int_{|z|>1} \log|z|^k\ d\mu_{L_{f,s}} = k\int_{|z|>1} \log|z|\ d\mu_{L_{f,s}}.$$

In other words, both sides of (8.1) scale upon replacing f by a power and so we are free to assume that the image of every vertex is fixed by f and the set-up in Section 3.3 applies.

Again applying Theorems 3.2 and 3.3 and Lemma 7.1 (1), we see that the L^2–torsion $-\rho^{(2)}(G_\phi)$ can be expressed in the following way for any $k \geq 1$:

$$\begin{aligned} -\rho^{(2)}(G_\phi) &= -\frac{1}{k}\rho^{(2)}(G_{\phi^k}) = \frac{1}{k}\sum_{s\in\mathcal{EG}(f)} \log\det{}_{G_{\phi^k}}(I - L_{f^k,s}) \\ &= \frac{1}{k}\sum_{s\in\mathcal{EG}(f)} \log\det{}_{G_\phi}(I - \iota_* L_{f^k,s}) = \frac{1}{k}\sum_{s\in\mathcal{EG}(f)} \log\det{}_{G_\phi}(I - L^k_{f,s}). \end{aligned}$$

Thus, by Theorem 4.1 (3), using the function $h(z) = z^k$, we have for each $k \geq 1$ that:

$$-\rho^{(2)}(G_\phi) = \frac{1}{k}\sum_{s\in\mathcal{EG}(f)} \int_{\mathbb{C}} \log\left|1 - z^k\right| d\mu_{L_{f,s}}.$$

We claim that for each $s \in \mathcal{EG}(f)$ that:

$$\lim_{k\to\infty} \frac{1}{k} \int_{\mathbb{C}} \log\left|1-z^k\right| \, d\mu_{L_{f,s}} = \int_{1<|z|} \log|z| \, d\mu_{L_{f,s}}. \tag{8.2}$$

To verify (8.2), fix an index $s \in \mathcal{EG}(f)$. Let f_s denote the restriction of f to the f–invariant connected subgraph $\Gamma'_s \subseteq \Gamma$. There is a corresponding free-by-cyclic subgroup $G_{\phi,s} \subseteq G_\phi$. Let H denote the f_s–invariant subgraph $\Gamma'_s \cap \Gamma_{s-1} \subseteq \Gamma'_s$. For the natural inclusion $\iota' \colon G_{\phi,s} \to G_\phi$, as $L_{f,s} = \iota'_* L_{f_s,H}$, we have $\mu_{L_{f,s}} = \mu_{L_{f_s,H}}$ by Lemma 7.1 (2). Hence it suffices to verify (8.2) using the measure $\mu_{L_{f_s,H}}$. To simplify notation, denote we denote the operator $L_{f_s,H}$ by L for the remainder.

Let $\lambda > 1$ be the constant from Theorem 7.4 applied to the homotopy equivalence $f_s \colon \Gamma'_s \to \Gamma'_s$ which satisfies the chain flare condition relative to the graph H by assumption. Fix $1 < \nu < \lambda$. We decompose the integral along the circles $|z| = \nu^{-1}$ and $|z| = \nu$ as follows:

$$\frac{1}{k} \int_{\mathbb{C}} \log\left|1-z^k\right| \, d\mu_L = \frac{1}{k} \int_{|z|<\nu^{-1}} \log\left|1-z^k\right| \, d\mu_L + \frac{1}{k} \int_{\nu^{-1}\le|z|\le\nu} \log\left|1-z^k\right| \, d\mu_L + \frac{1}{k} \int_{\nu<|z|} \log\left|1-z^k\right| \, d\mu_L.$$

We treat these three integral separately.

By Theorems 4.4, 7.3 and 7.4, we have:

$$\int_{\nu^{-1}\le|z|\le\nu} \log\left|1-z^k\right| \, d\mu_L = \log\det{}_{G_{\phi,s}}(I-L^k)\Big|_{W^{(2)}_{\mathrm{qf}}} = 0.$$

Notice that for all $z \in \mathbb{C}$ with $|z| \le \nu^{-1}$, we have:

$$\lim_{k\to\infty} \left|1-z^k\right|^{1/k} = 1 \quad \text{and} \quad \left|\log\left|1-z^k\right|^{1/k}\right| \le \left|\log(1-\nu^{-1})\right|.$$

As constant functions are μ_L–measurable since $\mu_L(\mathbb{C}) < \infty$ (Theorem 4.1 (2)), by the Lebesgue dominated convergence theorem we find:

$$\frac{1}{k} \int_{|z|<\nu^{-1}} \log\left|1-z^k\right| \, d\mu_L = \int_{|z|<\nu^{-1}} \log\left|1-z^k\right|^{1/k} \, d\mu_L \to 0, \text{ as } k \to \infty.$$

Likewise, let $r = \max\{\|L\|, \nu\} + 1$ and thus $\mu_L(\{z \in \mathbb{C} \mid |z| > r\}) = 0$ by Theorem 4.1 (1). For all $z \in \mathbb{C}$ with $\nu \le |z| \le r$, we have:

$$\lim_{k\to\infty} \left|1-z^k\right|^{1/k} = |z| \quad \text{and} \quad \left|\log\left|1-z^k\right|^{1/k}\right| \le \max\{|\log(\nu-1)|, \log(1+r)\}.$$

Hence, by the Lebesgue dominated convergence theorem again, we find

$$\frac{1}{k} \int_{\nu<|z|} \log\left|1-z^k\right| \, d\mu_L = \int_{\nu<|z|<r} \log\left|1-z^k\right|^{1/k} \, d\mu_L \to \int_{\nu<|z|<r} \log|z| \, d\mu_L = \int_{\nu<|z|} \log|z| \, d\mu_L \text{ as } k \to \infty.$$

Hence, combining these three calculations we have

$$\lim_{k\to\infty} \frac{1}{k} \int_{\mathbb{C}} \log\left|1-z^k\right|^{1/k} \, d\mu_L = \int_{\nu<|z|} \log|z| \, d\mu_L$$

for all $1 < \nu < \lambda$ and so (8.2) holds. Thus

$$-\rho^{(2)}(G_\phi) = \lim_{k\to\infty} \sum_{s\in\mathcal{EG}(f)} \frac{1}{k} \int_{\mathbb{C}} \log \left|1 - z^k\right|^{1/k} \, d\mu_{L_{f,s}} = \sum_{s\in\mathcal{EG}(f)} \int_{1<|z|} \log|z| \, d\mu_{L_{f,s}}$$

as desired verifying (8.1).

It remains to show that each of the integrals in (8.1) is positive. As the operator L is given by right-multiplication by a matrix with coefficients in $\mathbb{Z}[G_{\phi,s}]$, we have $\log\det_{G_{\phi,s}}(L) \geq 0$ according to [25, Theorem 13.3 (2) and Lemma 13.11 (4)]. Therefore, we find

$$0 \leq \log\det_{G_{\phi,s}}(L) = \int_{\mathbb{C}} \log|z| \, d\mu_L = \int_{|z|<1} \log|z| \, d\mu_L + \int_{1<|z|} \log|z| \, d\mu_L.$$

As $\log|z| < 0$ when $|z| < 1$ and $0 < \log|z|$ when $1 < |z|$, if

$$\int_{1<|z|} \log|z| \, d\mu_L = 0$$

then $\mu_L(\{z \in \mathbb{C} \mid |z| \neq 1\}) = 0$. However by Theorem 4.1 (2) we have $\mu_L(\mathbb{C}) = n_s \geq 2$ and by [20, Main Theorem 1.1], Proposition 7.2 and Theorem 7.4 we have $\mu_L(\{z \in \mathbb{C} \mid |z| = 1\}) = \dim_{G_{\phi,s}}(W^{(2)}_{\mathrm{qf}}) \leq 1$. Hence, $\mu_L(\{z \in \mathbb{C} \mid |z| \neq 1\}) \neq 0$ showing that the integral is indeed positive. □

9. Applying the chain flare condition

In this final section, we include some final remarks about the chain flare condition. First, we explain how a CT representative $f\colon \Gamma \to \Gamma$ can be used to find a natural candidate for the quasi-fixed submodule V_{qf} associated the homotopy equivalence $f_s\colon \Gamma'_s \to \Gamma'_s$ that satisfies (CFH3). We will not give the complete definition of a CT (completely split relative train-track map), but only recall the properties we require as needed. See the works by Feighn–Handel [15] and Handel–Mosher [21] for full details on CTs. Second, we present an example for consideration in which the chain flare condition simplifies. Lastly, we include some remarks about applying the techniques of this paper to ascending HNN-extensions over free groups.

9.1. **A candidate for the quasi-fixed submodule.** First, we recall the notion of Nielsen path from which the notion of Nielsen 1–chain is modeled. A non-trivial edge-path γ in Γ is a *Nielsen path* if $f(\gamma)$ is homotopic rel endpoints to γ. In particular, the endpoints of a Nielsen path γ are fixed. We say the Nielsen path ρ is *closed* if the endpoints are the same. In a CT, the endpoints of a Nielsen path are vertices [15, Definition 4.7 (4)].

Suppose that $f\colon \Gamma \to \Gamma$ is a CT with respect to the filtration $\emptyset = \Gamma_0 \subset \Gamma_1 \subset \cdots \subset \Gamma_S$. By definition, the filtration is reduced [15, Definition 4.7 (3)]. Fix an index $s \in \mathcal{EG}(f)$ and let f_s denote the restriction of f to the connected subgraph Γ'_s and let $\mathbb{F}_s$ denote the subgroup (well-defined up to conjugacy) determined by Γ'_s. Let H denote the f_s–invariant subgraph $\Gamma'_s \cap \Gamma_{s-1} \subset \Gamma'_s$. Up to reversal of orientation, there is at most one Nielsen path contained in Γ'_s that is not contained in H [15, Corollary 4.19 eg-(i)].

If there is no such Nielsen path, we set $V_{\mathrm{qf}} = \{0\}$. Else, let γ_s be the Nielsen path and let $* \in \mathsf{V}(\Gamma'_s)$ be one the of the endpoints of γ_s. We fix a lift $\tilde{*} \in \mathsf{V}(\widetilde{\Gamma}'_s)$ of $* \in \mathsf{V}(\Gamma'_s)$ and a lift $\tilde{f}_s\colon \widetilde{\Gamma}'_s \to \widetilde{\Gamma}'_s$ of $f_s\colon \Gamma'_s \to \Gamma'_s$ that fixes $\tilde{*}$. There is a lift $\widetilde{\gamma}_s$ of γ_s to $\widetilde{\Gamma}'_s$ so that one of its endpoints is $\tilde{*}$; let $v \in \mathsf{V}(\widetilde{\Gamma}'_s)$ be the other endpoint of $\widetilde{\gamma}_s$. As γ_s is fixed up to homotopy rel endpoints by f_s, we have $\tilde{f}_s(v) = v$. We claim that $\rho = \pi^{\perp}_H([\tilde{*}, v]) \in C_1(\widetilde{\Gamma}'_s; \widetilde{H}; \mathbb{Q})$ is a Nielsen 1–chain. To this end, there are two cases depending on whether or not γ_s is closed.

If the Nielsen path γ_s is not closed, then there is an edge $e \in \mathsf{E}(\Gamma'_s) - \mathsf{E}(H)$ is is crossed exactly once by γ_s [21, Fact 1.42 (1)]. The lift of this edge to $\widetilde{\Gamma}'_s$ contained in $\widetilde{\gamma}_s$ satisfies item (NNC1) and so ρ is a non-geometric Nielsen–1–chain in this case. (In this case, the stratum corresponding to Γ'_s is a termed a non-geometric stratum.) We set V_{qf} to the $\mathbb{Q}[\mathbb{F}]$–submodule generated by ρ.

If γ_s the Nielsen path is closed, then every edge in $\mathsf{E}(\Gamma'_s) - \mathsf{E}(H)$ is crossed exactly twice by γ_s [21, Fact 1.42 (2)]. Moreover, there exists a weak geometric model for $f_s \colon \Gamma'_s \to \Gamma'_s$ [21, Fact 2.3]. This includes the following data [21, Definition 2.1]:

(1) a compact connected surface Σ with negative Euler characteristic and non-empty boundary whose components are $\partial\Sigma = \partial_0\Sigma \cup \cdots \cup \partial_m\Sigma$;
(2) a 2–complex Y that is the quotient of attaching Σ to Γ_{s-1} via given homotopically nontrivial maps $\partial_j\Sigma \to \Gamma_{s-1}$, $j = 1, \ldots, m$;
(3) an embedding $\Gamma'_s \hookrightarrow Y$ extending the embedding $\Gamma_{s-1} \hookrightarrow Y$ where $Y - (\Gamma'_s \cup \partial_0\Sigma)$ is an open 2–disc; and
(4) the boundary component $\partial_0\Sigma$ is homotopic in Y to γ_s.

In this case, we claim that ρ is a geometric Nielsen–1–chain. We verify the three items in turn.

Firstly, suppose $g \in \mathbb{F}_s$ is nontrivial and $\mathrm{supp}(\rho) \cap \mathrm{supp}(g\rho)$ is non-empty. Let β be the maximal subedge-path of the edge-path $\widetilde{\gamma}_s$ such that $g\beta \subseteq \widetilde{\gamma}_s$. Suppose there are two edges in β that lie in $\mathsf{E}(\widetilde{\Gamma}'_s) - \mathsf{E}(\widetilde{H})$. Taking an innermost pair, there is a subedge-path of β of the form $e_1 \cdot \beta' \cdot e_2$ where $e_1, e_2 \in \mathsf{E}(\widetilde{\Gamma}'_s) - \mathsf{E}(\widetilde{H})$ and β' is a (possibly trivial) edge-path in $\widetilde{H}$. If β' is trivial, the vertex corresponding to β' has valence two as Σ is a surface. However, in a CT, we can always arrange that the only vertices of valence two in Γ'_s either lie in Γ_{s-1} or else are an endpoint of a Nielsen path. If β' is non-trivial, then its image in $\Gamma_{s-1} \subset Y$ corresponds to one of the boundary components $\partial_j\Sigma$ for some $j = 1, \ldots, m$. In this case, we see that the endpoints of β' have the same image in Y and the link of this vertex in Σ is disconnected. This is a contradiction as Σ is a surface. Therefore we see that $\mathrm{supp}(\rho) \cap \mathrm{supp}(g\rho)$ can contain at most one edge of $\mathsf{E}(\widetilde{\Gamma}'_s) - \mathsf{E}(\widetilde{H})$, verifying (GNC1).

Next, as every edge in in $\mathsf{E}(\Gamma'_s) - \mathsf{E}(H)$ is crossed exactly twice by γ_s, we see that (GNC2) holds.

Lastly, using the language of Section 5.2, let $T_0 \subseteq T_\rho$ be the component that contains $\mathrm{id}_{\mathbb{F}_s}$. The stabilizer of T_0 is the subgroup $\pi_1(\Sigma) \subseteq \mathbb{F}_s$, which is well-defined up to conjugacy. As this subgroup acts transitively on the vertices of T_0, we have that the elements of $\pi_1(\Sigma)$ corresponding to the vertices adjacent to $\mathrm{id}_{\mathbb{F}_s}$ generate $\pi_1(\Sigma)$. As Σ has negative Euler characteristic, $\pi_1(\Sigma)$ is non-abelian and hence at least two of these elements do not commute, verify (GNC3).

Hence, ρ is a geometric Nielsen–1–chain in this case. (In this case, the stratum corresponding to Γ'_s is termed a geometric stratum.) We set V_{qf} to the $\mathbb{Q}[\mathbb{F}]$–submodule generated by ρ.

As every outer automorphism $\phi \in \mathrm{Out}(\mathbb{F})$ has power that is represented by a CT [15, Theorem 4.28, Lemma 4.42], we see that up to replacing ϕ by an iterate, we have a natural choice for V_{qf} for which (CFH3) holds.

Remark 9.1. In order to show that $-\rho^{(2)}(G_\phi) > 0$ for all $\phi \in \mathrm{Out}(\mathbb{F})$ that are fully irreducible, it suffices to assume that $H \subseteq \Gamma$ is a single vertex (hence can be ignored) and that $V_{\mathrm{qf}} = \{0\}$. Indeed, if a CT map for ϕ has a Nielsen path, then the stable tree T_ϕ is geometric (as an $\mathbb{R}$–tree) [2, Theorem 3.2]. Hence if CT maps for both ϕ and ϕ^{-1} contain Nielsen paths, then both T_ϕ and $T_{\phi^{-1}}$ are geometric. According to [19, Corollary 9.3], this implies that ϕ is induced by a pseudo-Anosov homeomorphism of a surface and hence $-\rho^{(2)}(G_\phi) > 0$ as it the volume of the corresponding mapping torus. Therefore, we may assume that a CT map for ϕ or ϕ^{-1} does not have any Nielsen paths and since $G_\phi = G_{\phi^{-1}}$, it suffices to work with the corresponding outer automorphism.

9.2. **An example to consider.** We formulated the chain flare condition in generality so that it could apply to a general free-by-cyclic group, where we represent the monodromy ϕ by a CT map. However, there are many free-by-cyclic groups for which the chain flare condition has a simpler form. For instance, consider the following automorphism of the free group of rank 3, $\mathbb{F} = \langle x_1, x_2, x_3 \rangle$, given by:

$$\Phi(x_1) = x_2, \quad \Phi(x_2) = x_3, \text{ and } \Phi(x_3) = x_1 x_2.$$

The obvious topological representative of Φ on the 3–rose $f\colon R_3 \to R_3$ is an irreducible train-track map with no Nielsen paths (same for Φ^{-1}) and so one would set $H = *$ (the unique vertex of R_3), $V_{\text{qf}} = \{0\}$ and $V_{\text{h}} = C_1(\widetilde{R}_3; \mathbb{Q})$. In this case, $A_f\colon C_1(\widetilde{R}_3; \mathbb{Q}) \to C_1(\widetilde{R}_3, \mathbb{Q})$ is a vector space isomorphism and one may verify the chain flare condition for an appropriate power by exhibiting constants $\lambda > 1$ and $N \geq 1$ such that for any integral 1–chain $x \in C_1(\widetilde{R}_3; \mathbb{Z})$ we have:

$$\lambda \|x\| \leq \max \left\{ \left\| A_f^N(x) \right\|, \left\| A_f^{-N}(x) \right\| \right\}.$$

9.3. **Ascending HNN-extensions.** An ascending HNN-extension of a free group $\mathbb{F}$ is the group given by a presentation:

$$\mathbb{F} *_\Psi = \langle \mathbb{F}, t \mid t^{-1} x t = \Phi(x) \text{ for } x \in \mathbb{F} \rangle$$

where $\Psi\colon \mathbb{F} \to \mathbb{F}$ is an injective endomorphism. Beyond being generalizations of free-by-cyclic groups, ascending HNN-extensions arise naturally in the study of free-by-cyclic groups and the study of injective endomorphisms has seen increased interest lately [12, 29, 31].

The discussion in Section 3.3—in particular Theorem 3.3—holds for ascending HNN-extensions. Where the similarity breaks down and further analysis is necessary is in Section 7. The key distinction between the free-by-cyclic and ascending HNN-extension cases lies in the way that $L^2(\mathbb{F})$ sits inside $L^2(\mathbb{F} \rtimes_\Phi \langle t \rangle)$ versus the way it sits inside $L^2(\mathbb{F} *_\Psi)$ when Ψ is non-surjective. In both cases, the relevant Hilbert space is the closure of the direct sum of a number of copies of $L^2(\mathbb{F})$, indexed by coset representative of $\mathbb{F}$ in the relevant group. In the free-by-cyclic case, we have that the operator $L_{f,H}$ respects this direct sum decomposition as written in (7.1), whereas this is not true in the ascending HNN-extension case when Ψ is non-surjective. This poses a problem in the proof of Theorem 7.4. One can create examples for $n \geq 1$ of the form $\xi = t^{-1}\xi^{(1)} + gt^{-1}\xi^{(2)} \in L^2(\mathbb{F} *_\Psi)$ where $\xi^{(1)}, \xi^{(2)} \in L^2(\mathbb{F})$ and $g \in \mathbb{F} - \Psi(\mathbb{F})$ such that $\left\| A_{f,H}^k(\xi^{(1)}) \right\|, \left\| A_{f,H}^k(\xi^{(2)}) \right\| \approx 2^k$ and yet $2^{-n} \|L_{f,H}(\xi)\| \leq \|\xi\|$. See the schematic in Figure 9 contrasting the two settings.

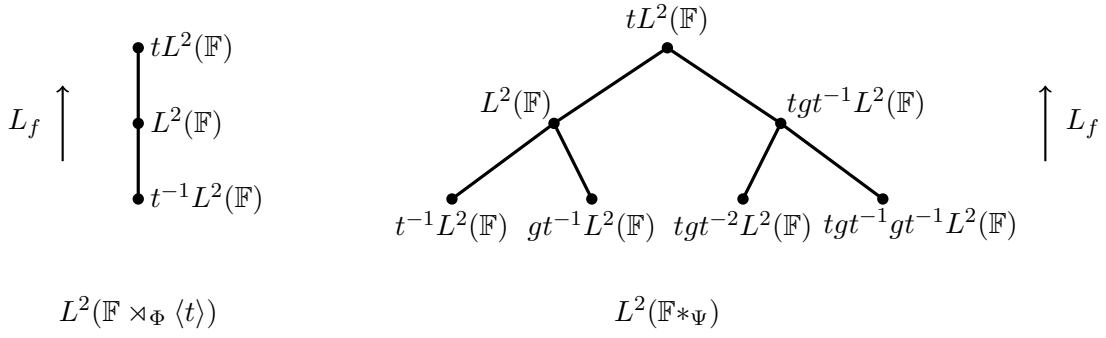

FIGURE 9. Contrasting the free-by-cyclic setting with the ascending HNN-extension setting.

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