

BOUNDED PROJECTIONS TO THE \mathcal{Z} -FACTOR GRAPH

MATT CLAY AND CAGLAR UYANIK

ABSTRACT. Suppose G is a free product $G = A_1 * A_2 * \cdots * A_k * F_N$, where each of the groups A_i is torsion-free and F_N is a free group of rank N . Let \mathcal{O} be the deformation space associated to this free product decomposition. We show that the diameter of the projection of the subset of \mathcal{O} where a given element has bounded length to the \mathcal{Z} -factor graph is bounded, where the diameter bound depends only on the length bound. This relies on an analysis of the boundary of G as a hyperbolic group relative to the collection of subgroups A_i together with a given non-peripheral cyclic subgroup. The main theorem is new even in the case that $G = F_N$, in which case \mathcal{O} is the Culler–Vogtmann outer space. In a future paper, we will apply this theorem to study the geometry of free group extensions.

1. INTRODUCTION

Let F be a finitely generated free group. Given a subgroup $H \subseteq \text{Out}(F)$, there is an extension of F by H , denoted E_H , obtained via the following diagram

$$\begin{array}{ccccccc} 1 & \longrightarrow & F & \longrightarrow & \text{Aut}(F) & \xrightarrow{p} & \text{Out}(F) \longrightarrow 1 \\ & & \parallel & & \uparrow & & \uparrow \\ 1 & \longrightarrow & F & \longrightarrow & E_H & \longrightarrow & H \longrightarrow 1 \end{array} \quad (1.1)$$

where $E_H = p^{-1}(H)$. Characterizing the subgroups $H \subseteq \text{Out}(F_N)$ such that the extension E_H is hyperbolic is an open question in geometric group theory. A characterization of hyperbolicity for these types of free group extension allows one to determine when any free group extension is hyperbolic [13, Section 2.5].

There is a long history behind this problem starting with Thurston’s work on 3-manifolds that fiber over the circle [32]. For a surface Σ of genus $g \geq 2$, the short exact sequence in the top row of (1.1) is the well known Birman exact sequence [3]:

$$1 \longrightarrow \pi_1(\Sigma, *) \longrightarrow \text{Mod}^\pm(\Sigma, *) \xrightarrow{p} \text{Mod}^\pm(\Sigma) \longrightarrow 1.$$

As for subgroups of $\text{Out}(F)$, every subgroup $H < \text{Mod}^\pm(\Sigma)$ gives rise to an extension $E_H = p^{-1}(H)$ of $\pi_1(\Sigma, *)$. The seminal result of Thurston alluded to above implies that when $H = \langle f \rangle$ is infinite cyclic the extension E_H is hyperbolic precisely when f is pseudo-Anosov. This characterization was extended to finitely generated subgroups H by combined work of Farb–Mosher [14] and Hamenstädt [20]. Specifically, their work shows that E_H is hyperbolic if and only if H is finitely generated and the orbit map into the Teichmüller space has quasi-convex image. Such a subgroup H is called *convex co-compact*. Every convex co-compact subgroup is itself hyperbolic and purely pseudo-Anosov, meaning that every infinite order element is pseudo-Anosov [14]. Hamenstädt [20] and Kent–Leininger [25] independently proved that a subgroup $H \subseteq \text{Mod}^\pm(\Sigma)$ is convex co-compact if and only if H is finitely generated and the orbit map $H \rightarrow \mathcal{C}(\Sigma)$ into the curve complex is a quasi-isometric embedding. Hence, for a finitely generated subgroup $H \subseteq \text{Mod}^\pm(\Sigma)$, an extension E_H is hyperbolic if and only if the orbit map $H \rightarrow \mathcal{C}(\Sigma)$ is a quasi-isometric embedding.

The situation in the setting of free group extensions has some partial progress analogous to the setting of surfaces, but a full characterization is still not known. There are two distinct analogs of pseudo-Anosov maps for free groups that play unique roles in the theory: fully irreducible elements and atoroidal elements. An element $\varphi \in \text{Out}(F)$ is called *fully irreducible* if no power of φ fixes the conjugacy class of a proper free factor of F . These are elements of $\text{Out}(F)$ that act on the closure of the Outer space with North-South dynamics [27], and are precisely the elements of $\text{Out}(F)$ that act as loxodromic isometries on the free factor graph [2]. An element $\varphi \in \text{Out}(F)$ is called *atoroidal* if no power of φ fixes the conjugacy class of a non-trivial element in F . Brinkmann proved that when $H = \langle \varphi \rangle$ is infinite cyclic, the extension E_H is hyperbolic precisely when φ is atoroidal [6].

Dowdall–Taylor were the first to produce a sufficient dynamical criterion on finitely generated subgroups $H \subseteq \text{Out}(F)$ that ensures hyperbolicity of E_H . Specifically, they prove that if H is finitely generated and the orbit map $H \rightarrow \text{CoS}(F)$ into the co-surface graph is a quasi-isometric embedding, then E_H is hyperbolic [12]. The co-surface graph (an electrification of the free factor graph) is hyperbolic and the loxodromic isometries of $\text{CoS}(F)$ are precisely the fully irreducible and atoroidal elements [12]. Hence the existence of a quasi-isometric embedding also implies that H is itself hyperbolic, and every infinite order element is both fully irreducible and atoroidal. The converse of the theorem by Dowdall–Taylor does not hold, even for infinite cyclic subgroups. In essence, the property of being atoroidal is a mixing condition on the conjugacy classes of elements of F whereas the property of being fully irreducible is a mixing condition on conjugacy classes of free factors of F . There exists atoroidal but not fully irreducible $\varphi \in \text{Out}(F)$ in all ranks at least 4, and for such $E_{\langle \varphi \rangle}$ is hyperbolic but the orbit map $\langle \varphi \rangle \rightarrow \text{CoS}(F)$ has bounded image.

Hence, to produce a more robust condition for hyperbolicity of free group extensions, one must keep track of the invariant free factors and focus on the dynamics relative to these subgroups. We make a first step towards this in this paper. Since we are only concerned with the dynamics relative to a collection of subgroups and not any dynamics within them, we do not necessarily only focus on free groups and can work in a more general setting. To this end, in this work we consider free products of the form

$$G = A_1 * \cdots * A_k * F_N$$

where F_N is a free group of rank N . Non-trivial elements or subgroups of G that are conjugate into one of the A_i 's are called *peripheral*. The set of G -conjugacy classes of the factors A_i is denoted by \mathcal{A} .

Associated to the pair (G, \mathcal{A}) are two spaces of trees that are the focus of this paper. One of these is the *relative outer space* $\mathcal{O} = \mathcal{O}(G, \mathcal{A})$, also known as a *deformation space*. This space parametrizes actions of G on metric simplicial trees where each of the A_i fixes a unique vertex, the vertex stabilizers are either trivial or conjugate to one of the A_i , and the stabilizer of an edge is trivial. Such trees are called *Grushko trees*. In the case when $\mathcal{A} = \emptyset$, so that G is free, the space \mathcal{O} is the well-known Culler–Vogtmann outer space [10]. The relative outer space \mathcal{O} plays the role of the Teichmüller space for a closed surface Σ . The other space of interest for this paper is the *\mathcal{Z} -factor graph* $\mathcal{ZF} = \mathcal{ZF}(G, \mathcal{A})$. In the case where $\mathcal{A} = \emptyset$, this graph is quasi-isometric to the co-surface graph $\text{CoS}(F)$ mentioned above that was defined by Dowdall–Taylor [12, Proposition 4.4]. Vertices in \mathcal{ZF} are actions on simplicial trees where each of the A_i fixes a unique vertex and the stabilizer of an edge is either trivial, or cyclic and non-peripheral. Such trees are called *\mathcal{Z} -splittings*. The \mathcal{Z} -factor graph is one of a handful of hyperbolic graphs for free products that play the role of the curve complex for a finite-type surface. For a detailed description and exposition of the connections between many of these, see the work of Guirardel–Horbez [16, Section 2].

The exact definitions of the relative outer space \mathcal{O} and the \mathcal{Z} -factor graph \mathcal{ZF} appear in Section 2. For now, we observe that if one forgets the metric on a Grushko tree $T \in \mathcal{O}$, the resulting simplicial

tree is a \mathcal{Z} -splitting and hence a vertex in \mathcal{ZF} . Thus we have a well-defined projection map $\pi: \mathcal{O} \rightarrow \mathcal{ZF}$. This projection map plays the role in the context of surfaces of the systole map $\text{Sys}: \text{Teich}(\Sigma) \rightarrow \mathcal{C}(\Sigma)$, which assigns to a marked hyperbolic surface $X \in \text{Teich}(\Sigma)$ one of its shortest curves.

Loosely speaking, the main result of this paper states that the projection of the region of \mathcal{O} where a given non-peripheral element has bounded length has bounded diameter in \mathcal{ZF} , where the diameter bound depends only on the length bound. To state it precisely, we require the following notation. If $g \in G$ is non-peripheral and $T \in \mathcal{O}$ is a Grushko tree, then g has an invariant line in T called its *axis*, denoted T_g . By $|g|_T$ we denote the number of edges in a fundamental domain for the action of g on T_g . We call this quantity the *combinatorial length* of g . For a real number $L > 0$, we set $\mathcal{O}_L(g) = \{T \in \mathcal{O} \mid |g|_T \leq L\}$.

We can now state our main theorem.

Theorem 1.1. *Let (G, \mathcal{A}) be a non-sporadic torsion-free free product. For all $L > 0$, there is a $D > 0$ such that for any non-peripheral element $g \in G$, the diameter of $\pi(\mathcal{O}_L(g)) \subset \mathcal{ZF}$ is at most D .*

The notion of non-sporadic appears in Section 2. Theorem 1.1 is new even in the case where $\mathcal{A} = \emptyset$ and thus G is free.

As alluded to earlier, Theorem 1.1 will be applied in a subsequent paper to provide a new sufficient condition for the hyperbolicity of the extension E_H generalizing the one due to Dowdall–Taylor. Essentially, Theorem 1.1 implies that if the orbit map along a geodesic in the subgroup H to \mathcal{ZF} makes definite progress, then no element of F can stay short for long while moving along corresponding fiber. This type of statement will be promoted to an “annuli flaring” condition ensuring hyperbolicity of E_H .

For the remainder of this introduction, we will give an outline of the proof of Theorem 1.1 and of the rest of the paper. In broad strokes, our proof is analogous to how one might prove the similar statement in the setting of surfaces concerning the systole map $\text{Sys}: \text{Teich}(\Sigma) \rightarrow \mathcal{C}(\Sigma)$. We will state this theorem for surfaces (which is known but does not appear in the literature), give a sketch of a proof, and comment on its relation to the proof of Theorem 1.1.

Theorem 1.2. *Let Σ be a finite type surface. For all $L > 0$, there is a $D > 0$ such that for any curve γ on Σ , the diameter of $\text{Sys}(\text{Teich}_L(\gamma)) \subset \mathcal{C}(\Sigma)$ is at most D .*

There are three cases corresponding to the topological type of the curve $\gamma \subset \Sigma$: simple, non-filling, and filling. We will deal with these one at a time in parallel with the free product setting.

Simple: First, suppose that γ is simple. It follows from basic hyperbolic geometry that there is a constant C such that the geometric intersection number between γ and $\text{Sys}(X)$ is at most CL when $X \in \text{Teich}_L(\gamma)$. As the distance between simple closed curves in the curve complex is bounded above by their geometric intersection number, for $X \in \text{Teich}_L(\gamma)$, we have that the distance in the curve complex between γ and $\text{Sys}(X)$ is at most CL . Using the curve γ as a central point to measure distances, it now follows that the diameter of $\text{Sys}(\text{Teich}_L(\gamma))$ is at most $2CL$.

In the setting of free products, we also consider simple elements first. In this context, a non-peripheral element $g \in G$ is called *simple* if it is elliptic in a \mathcal{Z} -splitting where all edge stabilizers are trivial. Such a \mathcal{Z} -splitting is called a *free splitting*. There is a non-empty subset of \mathcal{ZF} with diameter equal to one consisting of all of the \mathcal{Z} -splittings in which g is elliptic. This subset plays the role of γ as a central point to which we will measure distance from for a tree in $\pi(\mathcal{O}_L(g))$. The main tool we exploit here is the ubiquitous notion of a *Whitehead graph*. Using an appropriate notion of a Whitehead graph for free-products due to Guirardel–Horbez (Section 4) and its properties for

simple elements, it will follow that if $T \in \mathcal{O}_L(g)$, then there is a free splitting $S \in \mathcal{ZF}$ for which g is elliptic and where $d(\pi(T), S) \leq L$. From this, we conclude that the diameter of $\pi(\mathcal{O}_L(g))$ is at most $2L + 1$ (Proposition 5.1).

Non-filling: Next, suppose that γ is contained in a proper subsurface $\Sigma_\gamma \subset \Sigma$ and let α be one of the boundary curves of Σ_γ . Then we see that $|\alpha|_X \leq |\gamma|_X$ for any $X \in \text{Teich}(\Sigma)$ and hence $\text{Teich}_L(\gamma) \subseteq \text{Teich}_L(\alpha)$ for any L . As α is simple, we have that the diameter of $\text{Sys}(\text{Teich}_L(\gamma))$ is bounded in terms of L by the first case.

In the setting of free products, the analogous elements are called \mathcal{Z} -simple. These are the elements $g \in G$ that are elliptic in some \mathcal{Z} -splitting. There are subcases here depending on whether or not the element g is *quadratic*, that is, if there is a geometric model for (G, \mathcal{A}) in which the conjugacy class of g corresponds to a boundary component (Definition 9.1). In both of these subcases, the key point is to try to find a \mathcal{Z} -splitting S where g is elliptic and where the length of an edge stabilizer in S is bounded by some function of the length of g . As edge stabilizers in \mathcal{Z} -splittings are simple, we can use the first case to get a bound on $\pi(\mathcal{O}_L(g))$ much like in the surface case.

However, the details in the two subcases are very different. In the quadratic case, we use the surface from the definition directly to find this \mathcal{Z} -splitting (Proposition 10.1). To this end, we provide a novel characterization of quadratic elements in terms of Whitehead graphs (Proposition 9.4) which generalizes the characterization due to Otal in the case when \mathcal{A} is empty [29, Theorem 2] (cf. [8, Theorem 6.1]).

In the non-quadratic case, we study a quotient of the boundary of G as a relatively hyperbolic group with respect to \mathcal{A} where we identify pairs of points parameterized by the conjugates of g . Such a space is called a *decomposition space*. This space is in fact the boundary of G as a hyperbolic group relative to a new collection of subgroups that includes \mathcal{A} . The study of the decomposition space employs techniques from and adds to the existing literature regarding using the boundary of a hyperbolic or a relatively hyperbolic group to understand \mathcal{Z} -splittings of the group as developed by Bowditch [4], Cashen–Macura [8], Cashen [7], Hallmark [21], and Hallmark–Hruska [22]. This analysis is carried out in Section 11 and the proof of Theorem 1.1 for general \mathcal{Z} -simple elements appears as Proposition 12.1. Prior to these sections, we must develop a finite model for working with decomposition spaces. This includes a generalization of the Guirardel–Horbez notion of a Whitehead graph that is not focused on a single vertex of a Grushko tree, but takes into account an entire locally finite subtree of a Grushko tree. This construction takes place in Section 7 and forms the basis for the analysis in Sections 9 and 11.

Filling: Finally, suppose that γ is a filling curve. Then the diameter of $\text{Teich}_L(\gamma)$ with the Teichmüller metric is bounded as a subset of $\text{Teich}(X)$. Since the systole map $\text{Sys}: \text{Teich}(\Sigma) \rightarrow \mathcal{C}(\Sigma)$ is coarsely Lipschitz, it follows that $\text{Sys}(\text{Teich}_L(\gamma))$ has bounded diameter. Up to homeomorphism of Σ , there are only finitely many curves on Σ that have length at most L on some hyperbolic surface X . Thus we can obtain a bound on the diameter of $\text{Sys}(\text{Teich}_L(\gamma))$ that depends only on L and not on γ .

In the setting of free products, we take a related, but different approach. Partly this is for efficiency, but partly this is also due to necessity. Indeed, it is not true that up to the action of outer automorphisms of G that preserve \mathcal{A} that there are only finitely many conjugacy classes of non-peripheral elements in G that have combinatorial length at most L in some Grushko tree T . Thus proving boundedness of $\pi(\mathcal{O}_L(g))$ for a given non-peripheral element g is not sufficient. To this end, we use contradiction and assume there is a sequence of elements (g_n) and sequences of Grushko trees (S_n) and (T_n) where $|g_n|_{S_n}, |g_n|_{T_n} \leq L$ but yet the distance between $\pi(S_n)$ and

$\pi(T_n)$ is unbounded. We show that with this set-up we can find a single non-peripheral element $g \in G$ where $|g|_{S_n}$ and $|g|_{T_n}$ are both bounded. This enables us to find a tree appearing in the closure of the relative outer space \mathcal{O} in which g is elliptic. As a consequence of the Rips machine, it follows that such an element g is necessarily \mathcal{Z} -simple. This is now a contradiction to the above case. This argument appears in Section 13.

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2. RELATIVE OUTER SPACE AND RELATIVE FACTOR GRAPHS

In this section, we introduce the basic setting and notation necessary for Theorem 1.1 and its proof.

Let A_1, \dots, A_k be countably infinite torsion-free groups and let G denote the free product

$$G = A_1 * \dots * A_k * F_N$$

where F_N is a free group of rank N . Any subgroup of G conjugate into one of A_1, \dots, A_k is called a *peripheral subgroup* and any element of G contained in a peripheral subgroup is called *peripheral* as well. The collection $\mathcal{A} = \{[A_1], \dots, [A_k]\}$ of conjugacy classes is called the *peripheral factor system* of G .

The *complexity* of (G, \mathcal{A}) is defined as the quantity $\xi(G, \mathcal{A}) = 2k + 3N - 3$. Throughout this paper we will always assume that $\xi(G, \mathcal{A}) \geq 3$. This excludes the following five cases which are called *sporadic*:

- $\xi(G, \mathcal{A}) = -3$: $G = \{1\}$ and $\mathcal{A} = \emptyset$
- $\xi(G, \mathcal{A}) = -1$: $G = A_1$ and $\mathcal{A} = \{[A_1]\}$
- $\xi(G, \mathcal{A}) = 0$: $G = \mathbb{Z}$ and $\mathcal{A} = \emptyset$
- $\xi(G, \mathcal{A}) = 1$: $G = A_1 * A_2$ and $\mathcal{A} = \{[A_1], [A_2]\}$
- $\xi(G, \mathcal{A}) = 2$: $G = A_1 * \mathbb{Z}$ and $\mathcal{A} = \{[A_1]\}$

The subgroup of $\text{Out}(G)$ consisting of outer automorphisms φ such that $\varphi([A_i]) = [A_i]$ for each $i = 1, \dots, k$ is denoted $\text{Out}(G, \mathcal{A})$.

A (G, \mathcal{A}) -tree is a metric simplicial tree T equipped with a minimal cocompact action of G by isometries without inversions such that each peripheral subgroup of G fixes a point in T . Two (G, \mathcal{A}) -trees are considered equivalent if there is a G -equivariant isometry between them. We will (almost) always assume that if some vertex of a (G, \mathcal{A}) -tree has degree two, then there is an element in G that fixes this vertex and interchanges the two incident edges.

For a point $p \in T$ in a (G, \mathcal{A}) -tree, a *direction at p* is a connected component of $T - \{p\}$. We will also refer to a connected component $Y \subset T - X$ as a *direction at X* where $X \subset T$ is a connected closed subset. There is a unique point $\partial_0 Y \in X$ such that Y is a direction at $\partial_0 Y$.

Given (G, \mathcal{A}) -trees T and T' , a *collapse* is a continuous G -equivariant function $f: T \rightarrow T'$ such that the pre-image of any point in T' is a connected subset of T . Two (G, \mathcal{A}) -trees T_0 and T_1 are *compatible* if there exists a (G, \mathcal{A}) -tree T and a pair of collapse maps $f_0: T \rightarrow T_0$ and $f_1: T \rightarrow T_1$.

There is a right action of $\text{Out}(G, \mathcal{A})$ on the set of equivalence classes of (G, \mathcal{A}) -trees. Namely, given $\varphi \in \text{Out}(G, \mathcal{A})$ one first fixes an automorphism $\Phi \in \text{Aut}(G)$ that represents φ . Then given a (G, \mathcal{A}) -tree T with action homomorphism $\rho: G \rightarrow \text{Isom}(T)$, we define a new action on T by the homomorphism $\rho_\Phi = \rho \circ \Phi$. It is straightforward to check that this defines an action satisfying the

requirements of a (G, \mathcal{A}) -tree and that the equivalence class of the new (G, \mathcal{A}) -tree is independent of the choices made. We denote this new (G, \mathcal{A}) -tree by $T\varphi$.

A (G, \mathcal{A}) -tree T is called a *Grushko tree* if the following conditions hold:

- (1) every elliptic element for T is peripheral, and
- (2) the stabilizer of any edge in T is trivial.

We observe that this implies that each peripheral subgroup fixes a unique vertex of T and that if v is a vertex of T whose stabilizer, $\text{Stab}_T(v)$ is non-trivial, then $\text{Stab}_T(v)$ is conjugate to one of the A_i .

If T is a Grushko tree, then the number of edges in T/G is at most the complexity $\xi(G, \mathcal{A})$, when (G, \mathcal{A}) is non-sporadic. Indeed, the maximum occurs precisely when the vertices with non-trivial stabilizer have degree one and each other vertex has degree three.

Definition 2.1. The set of equivalence classes of Grushko trees for (G, \mathcal{A}) is called the *relative outer space* and is denoted $\mathcal{O} = \mathcal{O}(G, \mathcal{A})$.

There is a natural way to equip this space with a topology where the $\text{Out}(G, \mathcal{A})$ -action is by homeomorphisms. See the work of Guirardel–Levitt for more details [17].

If $T \in \mathcal{O}$ is a Grushko tree for (G, \mathcal{A}) and $g \in G$ is non-peripheral, it acts on T as a hyperbolic isometry and by T_g we denote its axis. Further, by $|g|_T$ we denote the number of edges in a fundamental domain for g on T_g . In other words, $|g|_T$ is the number of edges in $T_g/\langle g \rangle$. For $g \in G$ non-peripheral and $L > 0$, we denote:

$$\mathcal{O}_L(g) = \{T \in \mathcal{O} \mid |g|_T \leq L\}.$$

We note that this set could possibly be empty.

A \mathcal{Z} -*splitting* of (G, \mathcal{A}) is a (G, \mathcal{A}) -tree where every edge has length one and the stabilizer of any given edge is either trivial or cyclic and non-peripheral. A non-peripheral element $g \in G$ is \mathcal{Z} -*simple* if there exists a \mathcal{Z} -splitting of (G, \mathcal{A}) where g is elliptic.

A \mathcal{Z} -splitting where the stabilizer of any given edge is trivial is called a *free-splitting* of (G, \mathcal{A}) . Notice that a Grushko tree where every edge has length one is a free-splitting. A non-peripheral element $g \in G$ is *simple* if there is a free-splitting of (G, \mathcal{A}) where g is elliptic.

We note that every simple element is also \mathcal{Z} -simple. Here is one way to construct \mathcal{Z} -simple elements that are not simple. Consider a free factorization of a finitely generated free group $F = A * B$ where A has rank at least two. Fix an element $a \in A$ that is not simple in A . Let S be the Bass–Serre tree of the amalgamated free product decomposition $A *_{\langle a \rangle} \langle a, B \rangle$; this is a \mathcal{Z} -splitting of F . Any element in $\langle a, B \rangle$ that is not simple in $\langle a, B \rangle$ is an example of a \mathcal{Z} -simple element of F as it is elliptic in S but is not simple in F , as can be seen using Whitehead graphs.

The following lemma of Horbez regarding edge stabilizers in \mathcal{Z} -splittings shows that the above construction is essentially the only way.

Lemma 2.2 ([23, Lemma 6.11]). *If g fixes an edge in a \mathcal{Z} -splitting, then g is simple.*

When $\mathcal{A} = \emptyset$ and thus G is a free group, this is a well-known fact originally due to Shenitzer [30] and Swarup [31].

Definition 2.3. The set of equivalence classes of \mathcal{Z} -splittings of (G, \mathcal{A}) form the vertex set of the \mathcal{Z} -*factor graph*, denoted $\mathcal{ZF} = \mathcal{ZF}(G, \mathcal{A})$. There is an edge joining the equivalence classes of two \mathcal{Z} -splittings T_0 and T_1 if either:

- (1) T_0 and T_1 are compatible, or
- (2) there exists a non-peripheral element $g \in G$ which is elliptic in both T_0 and T_1 .

We let d denote the path metric on the graph \mathcal{ZF} .

We note that for free groups, i.e., when $\mathcal{A} = \emptyset$, the \mathcal{Z} -factor graph is quasi-isometric to the Kapovich–Lustig intersection graph [24] and also to the Dowdall–Taylor co-surface graph [12]. See the work of Guirardel–Horbez for these facts and more information about the \mathcal{Z} -factor graph, including a proof of its hyperbolicity and a description of its boundary [16].

There is a natural $\text{Out}(G, \mathcal{A})$ -equivariant map $\pi: \mathcal{O} \rightarrow \mathcal{ZF}$ defined by scaling the metric on each edge to have length one.

3. BOUNDARIES AND LINES IN FREE PRODUCTS

As mentioned in the Introduction, central to the proof of Theorem 1.1 is an analysis of the \mathcal{Z} -splittings of a \mathcal{Z} -simple element. This is carried out by considering the free product (G, \mathcal{A}) as a relatively hyperbolic group, both with the structure given by the free factor system \mathcal{A} , and with the structure given by including the conjugacy class of the maximal cyclic subgroup containing a given non-peripheral element. Under these considerations, we relate \mathcal{Z} -splittings of the free product where a given non-peripheral element is elliptic, to cut sets in the corresponding boundary. This idea was first used to understand \mathcal{Z} -splittings of hyperbolic groups by Bowditch [4]. It was applied to relatively hyperbolic groups by Hallmark [21] and Hallmark–Hruska [22]. Preceding the work of Hallmark and Hallmark–Hruska in the general setting of relatively hyperbolic groups, is the work of Cashen–Macura [8] and Cashen [7]. These papers apply the above ideas to the case of a finitely generated free group, F , building on work of Otal [29], to understand the \mathcal{Z} -splittings of F where a given \mathcal{Z} -simple element is elliptic. By focusing on the setting of a free group, they are able to obtain the types of important quantitative bounds that we need for Theorem 1.1. We closely follow their analysis. For clarity, in places we opt for a direct argument in the setting of free products as opposed to appealing to a general statement regarding all relatively hyperbolic groups. In a different direction than what we pursue here, Barrett has been able to obtain quantitative bounds similar to those of Cashen–Macura in the setting of hyperbolic groups [1].

3.1. Boundaries. In this section, we introduce the various notions of boundary used in the sequel and talk about how they are related. To begin, we discuss the boundary of the pair (G, \mathcal{A}) as a free product as defined by Guirardel–Horbez [15].

Given a Grushko tree $T \in \mathcal{O}$, we define the following spaces:

$\partial_\infty T$ = the Gromov boundary of T , i.e., equivalence classes of rays $r: [0, \infty) \rightarrow T$

$\widehat{T} = T \cup \partial_\infty T$, with the usual topology on the union of a hyperbolic space and its boundary

$V_\infty(T)$ = the set of vertices of T with non-trivial stabilizer

$$\partial T = V_\infty(T) \cup \partial_\infty T \subset \widehat{T}_{\text{obs}}$$

In the last line, \widehat{T}_{obs} is the set $T \cup \partial_\infty T$ considered with the observers topology. A basis element for the observers topology consists of connected component (in the usual sense) of $\widehat{T} - P$, where $P \subset \widehat{T}$ is a finite set of points. In other words, a basis consists of intersections of finitely many directions. In particular, a sequence $(p_n) \subset T$ converges to $p_\infty \in \widehat{T}_{\text{obs}}$ in the observers topology if for all $q \in \widehat{T}_{\text{obs}} - \{p_\infty\}$, the direction at q that contains p_∞ also contains p_n for all but finitely many n . With the observers topology, \widehat{T}_{obs} is a compact space and $\partial T \subset \widehat{T}_{\text{obs}}$ is closed, hence also compact.

Note, a point $v \in V_\infty(T)$ can also be viewed as the set of rays $r: [0, \Delta] \rightarrow T$ where $r(\Delta) = v$.

As described by Guirardel–Horbez, for any two Grushko trees $T, T' \in \mathcal{O}$, there is a canonical G -equivariant homeomorphism $h: \partial T_{\text{obs}} \rightarrow \partial T'_{\text{obs}}$ such that $h(V_\infty(T)) = h(V_\infty(T'))$ and $h(\partial_\infty T) = \partial_\infty T'$ [15, Lemma 2.2]. Hence the free product (G, \mathcal{A}) has a well-defined boundary $\partial(G, \mathcal{A})$ and this boundary contains well-defined subsets $V_\infty(G, \mathcal{A})$ and $\partial_\infty(G, \mathcal{A})$.

Moreover, we have that $\partial(G, \mathcal{A})$ is G -equivariantly homeomorphic to the Bowditch boundary of G as a hyperbolic group relative to \mathcal{A} . Indeed, using the language and notation of Bowditch's work, T is a fine hyperbolic graph and the action of G on T satisfies the second definition of relative hyperbolicity with respect to the collection \mathcal{A} [5]. The induced topology on ∂T as a subset of \widehat{T}_{obs} defined above is exactly the topology described by Bowditch on ΔT , once vertices with trivial stabilizers are removed from ΔT . Such points are isolated. It is then shown by Bowditch that ΔT minus the isolated points is G -equivariantly homeomorphic to the boundary of (G, \mathcal{A}) as a relatively hyperbolic group [5, Proposition 9.1]. This connection was also observed by Knopf [26, Definition 2.2].

3.2. Lines. Recall that for a non-peripheral element $g \in G$ and a Grushko tree $T \in \mathcal{O}$, the axis of g is denoted T_g . This axis defines two points $T_g^+, T_g^- \in \partial_\infty T$ and consequently two points $g^\infty, g^{-\infty} \in \partial_\infty(G, \mathcal{A})$. These points are characterized by the fact that for any point $p \in T$ we have $\lim_{m \rightarrow \infty} g^m p = T_g^+$ and $\lim_{m \rightarrow \infty} g^{-m} p = T_g^-$. A two point set of the form $\{g^\infty, g^{-\infty}\} \subset \partial_\infty(G, \mathcal{A})$ is called a *periodic line*. Given a periodic line $\ell = \{g^\infty, g^{-\infty}\}$ and a Grushko tree $T \in \mathcal{O}$, by ℓ_T we denote the axis T_g .

If $g \in G$ is non-peripheral we set $\mathcal{L}_g = \{\{ag^\infty, ag^{-\infty}\} \mid a \in G\}$. In terms of a Grushko tree $T \in \mathcal{O}$, the lines in \mathcal{L}_g correspond to the endpoints of the axes $T_{aga^{-1}}$ for each $a \in G$. We observe that $\mathcal{L}_g = \mathcal{L}_{g^m}$ for any nonzero $m \in \mathbb{Z}$.

A *periodic line collection* is a collection of the form:

$$\mathcal{L} = \mathcal{L}_{g_1} \cup \cdots \cup \mathcal{L}_{g_m}$$

where each $g_1, \dots, g_m \in G$ is a non-peripheral element. We call the elements g_1, \dots, g_m the *generators* of the periodic line collection. For \mathcal{L} as above, we define

$$|\mathcal{L}|_T = \sum_{j=1}^m |g_j|_T.$$

In a free-product, every non-peripheral element $g \in G$ is contained in a unique maximal 2-ended subgroup denoted N_g . In fact N_g is both the normalizer of g in G and the stabilizer of the axis $T_g \subset T$. Thus, for a general free product, N_g is isomorphic to either \mathbb{Z} or to $\mathbb{Z}_2 * \mathbb{Z}_2$. As we are assuming that our free product (G, \mathcal{A}) is torsion-free, we must have that N_g is isomorphic to \mathbb{Z} . Given a periodic line collection $\mathcal{L} = \mathcal{L}_{g_1} \cup \cdots \cup \mathcal{L}_{g_m}$, we denote $\mathcal{N}_{\mathcal{L}} = \{[N_{g_1}], \dots, [N_{g_m}]\}$. When \mathcal{L} has a single generator g , we will use the notation \mathcal{N}_g . Note that this is just the conjugacy class of the subgroup $\langle g' \rangle$ where g' is an indivisible root of g .

3.3. Decomposition spaces. In the setting of free groups, decomposition spaces were first introduced by Otal to study free splittings of free groups are related items [29].

Definition 3.1. Suppose (G, \mathcal{A}) is a torsion-free free product and let \mathcal{L} be a periodic line collection. The *decomposition space* for \mathcal{L} , denoted $\mathcal{D}(\mathcal{L})$, is the quotient of $\partial(G, \mathcal{A})$ by the collection \mathcal{L} . In other words, if $\mathcal{L} = \mathcal{L}_{g_1} \cup \cdots \cup \mathcal{L}_{g_m}$, then we identify the points $ag_j^{-\infty} \sim ag_j^\infty$ for each $a \in G$ and $j = 1, \dots, m$. The quotient map is denoted by $q: \partial(G, \mathcal{A}) \rightarrow \mathcal{D}(\mathcal{L})$.

When necessary, we will write $\mathcal{D}_{(G, \mathcal{A})}(\mathcal{L})$ and $q_{(G, \mathcal{A})}$ if we need to keep track of the free product (G, \mathcal{A}) .

The connection between the decomposition space and the Bowditch boundary of a relatively hyperbolic group is well-known to experts and is given by the following lemma. For a proof, see the work of Hallmark–Hruska and the references within the proof of [22, Proposition 6.7].

Lemma 3.2. *Suppose (G, \mathcal{A}) is a torsion-free free product and that \mathcal{L} is a periodic line collection. Then G is hyperbolic relative to $\mathcal{A} \cup \mathcal{N}_{\mathcal{L}}$ and the Bowditch boundary of the relatively hyperbolic group $(G, \mathcal{A} \cup \mathcal{N}_{\mathcal{L}})$ is G -equivariantly homeomorphic to $\mathcal{D}(\mathcal{L})$.*

4. RELATIVE WHITEHEAD GRAPHS À LA GUIRADEL–HORBEZ

We seek to have a finite model for understanding the topology of a decomposition space $\mathcal{D}(\mathcal{L})$. As in the case when $\mathcal{A} = \emptyset$ studied by Otal [29] and Cashen–Macura [8], the object we ultimately use is a generalization of the classical notion of a Whitehead graph. To this end, in this section we introduce the concept of a Whitehead graph for Grushko trees in \mathcal{O} . In the setting of free products, the Whitehead graph of a non-peripheral element was defined by Guirardel–Horbez. As in the classical setting, Whitehead graphs can be used to detect if an element is simple. We state the version of this fact in this setting due to Guirardel–Horbez in Proposition 4.8.

Let (G, \mathcal{A}) be a fixed non-sporadic free product. Guirardel–Horbez define the notion of a Whitehead graph at a vertex v in some Grushko tree $T \in \mathcal{O}$ for a given periodic line collection \mathcal{L} , denoted by $\text{Wh}_T(\mathcal{L}, v)$ [15, Section 5.1]. We give a slight, but equivalent, variant of their definition that naturally leads to the generalization we define in Section 7. The vertex set of $\text{Wh}_T(\mathcal{L}, v)$ is the set of equivalence classes of directions at v , $Y \subset T - \{v\}$, under the action of $\text{Stab}_T(v)$. In each equivalence class, $[Y]$, a preferred direction is chosen; these preferred directions are enumerated by Y_1, \dots, Y_m . There is an oriented edge in $\text{Wh}_T(\mathcal{L}, v)$ from $[Y_i]$ to $[Y_j]$, labeled by $a \in \text{Stab}_T(v)$, for each line $\ell \in \mathcal{L}$ such that ℓ_T that meets both Y_i and aY_j . Note, in this case $a^{-1}\ell_T$ meets both Y_j and $a^{-1}Y_i$ and represents the same edge with the opposite orientation. We remark that this definition also makes sense for any point $p \in T$ that is not a vertex. In this case $\text{Wh}_T(\mathcal{L}, p)$ has exactly two equivalence classes, each of which consists of a unique direction. Further, in this case, the Whitehead graph $\text{Wh}_T(\mathcal{L}, p)$ is disconnected if and only if no geodesic in \mathcal{L} contains the point p .

Example 4.1. Let G be the free group of rank three with basis $\{a, b, c\}$ and we consider the free factor system $\mathcal{A} = \{[\langle a \rangle]\}$. Let $T \in \mathcal{O}$ be the Grushko tree where all edges have length 1 and T/G is the 2-rose where one edge is labeled by b and the other labeled by c . This quotient graph of groups is shown in Figure 1.

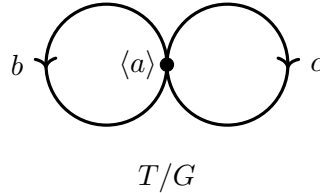


FIGURE 1. The quotient graph of groups T/G in Example 4.1.

Let v be the unique vertex of T fixed by a and let $g = bacb^{-1}a^3c^{-1}$. Let e_1 be the edge incident to v and bv and let e_2 be the edge incident to v and cv . We denote the direction at v that contains bv by Y_b^+ and we denote the direction at v that contains $b^{-1}v$ by Y_b^- . Similarly, we use Y_c^+ and Y_c^- to denote the directions at v that contain cv or $c^{-1}v$ respectively. The axes of g that meet

one of these four directions are shown on the left in Figure 2. Using these preferred directions, the Whitehead graph $\text{Wh}_T(\mathcal{L}_g, v)$ is shown on the right in Figure 2.

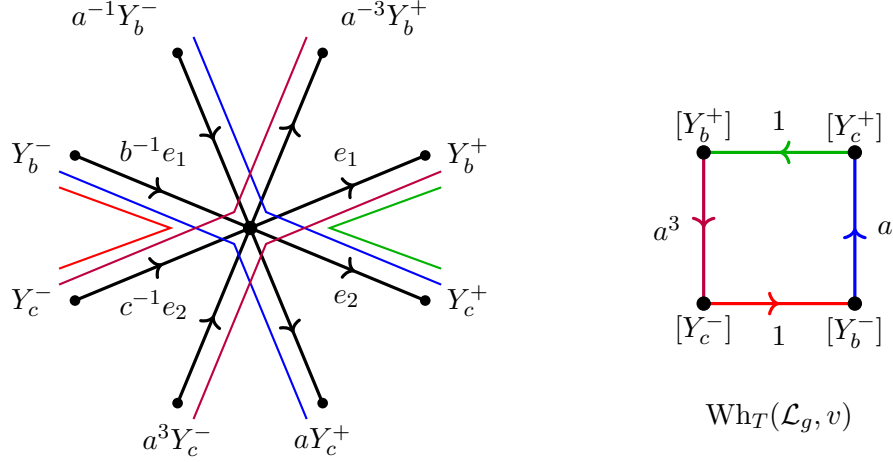


FIGURE 2. Some translates of T_g in T that meet v and the Whitehead graph $\text{Wh}_T(\mathcal{L}_g, v)$ from Example 4.1.

For a connected subgraph $U \subseteq \text{Wh}_T(\mathcal{L}, v)$ and a vertex $[Y] \in U$, the subgroup of $\text{Stab}_T(v)$ generated by the products of labels along the closed paths based at $[Y]$ is denoted $\text{Mon}(U, [Y])$ and is called the *monodromy subgroup*. Referring to Example 4.1, we have that $\text{Mon}(\text{Wh}_T(\mathcal{L}_g, v), [Y_c^-]) = \langle a^4 \rangle$.

As noted by Guirardel–Horbez, the conjugacy class of $\text{Mon}(U, [Y])$ in $\text{Stab}_T(v)$ is independent of the choice of vertex $[Y]$ and the choices of preferred representatives Y_1, \dots, Y_m . Moreover, given a maximal tree $\tau \subseteq U$, it is possible to choose preferred directions so that the label on each edge in τ is trivial. The conjugacy class of $\text{Mon}(U, [Y])$ in $\text{Stab}_T(v)$ is denoted $\text{Mon}(U)$. In the case that $\text{Mon}(U)$ is the conjugacy class of the trivial subgroup $\{1\}$ or the conjugacy class of $\text{Stab}_T(v)$, we will abuse notation and consider $\text{Mon}(U)$ as the subgroup $\{1\}$ or $\text{Stab}_T(v)$ respectively.

Definition 4.2. Let $T \in \mathcal{O}$ be a Grushko tree, let $v \in T$ be a vertex and let \mathcal{L} be a periodic line collection. The Whitehead graph $\text{Wh}_T(\mathcal{L}, v)$ has an *admissible cut* if either:

- (type i) $\text{Wh}_T(\mathcal{L}, v)$ is a disjoint union $U \sqcup V$ where:
 - U is connected with $\text{Mon}(U) = \{1\}$, and
 - V is nonempty if $\text{Stab}_T(v)$ is trivial; or
- (type ii) $\text{Wh}_T(\mathcal{L}, v)$ is a union $U \cup V$ where $U \cap V$ is a single vertex, U is connected and $\text{Mon}(U) = \{1\}$.

We remark that the second requirement for a type i admissible cut is omitted from the definition by Guirardel–Horbez. However, this requirement is implicit in their arguments involving the notion of an admissible cut. We record the following characterizations of admissible cuts. The proof of this lemma follows from the proof of Lemma 5.2 in the work of Guirardel–Horbez [15].

Lemma 4.3. Let $T \in \mathcal{O}$ be a Grushko tree, let $v \in T$ be a vertex and let \mathcal{L} be a periodic line collection.

- (1) The Whitehead graph $\text{Wh}_T(\mathcal{L}, v)$ has a type i admissible cut if and only if there is a Grushko tree $T' \in \mathcal{O}$ and a nontrivial collapse $T' \rightarrow T$ that is injective when restricted to $\ell_{T'}$ for any line $\ell \in \mathcal{L}$.

- (2) The Whitehead graph $\text{Wh}_T(\mathcal{L}, v)$ has a type ii admissible cut if and only if there is a Grushko tree $T' \in \mathcal{O}$ and a fold $T' \rightarrow T$ that is injective when restricted to $\ell_{T'}$ for any line $\ell \in \mathcal{L}$.

The definition of a fold is not used the sequel. The property that we will make use of is that if there is a fold $T' \rightarrow T$, then there is a Grushko tree T_0 and collapse maps $T_0 \rightarrow T$ and $T_0 \rightarrow T'$. In particular, we have that T and T' are compatible and thus $d(\pi(T), \pi(T')) \leq 1$.

Lemma 4.4. *Let $T \in \mathcal{O}$ be a Grushko tree and let $g \in G$ be a non-peripheral element. If either,*

- (1) *there is an edge $e \in T$ that is not crossed by aT_g for any $a \in G$, or*
- (2) *there is a vertex $v \in T$ such that $\text{Wh}_T(\mathcal{L}_g, v)$ has a type i admissible cut,*

then g is simple. Moreover, there is a free splitting S for (G, \mathcal{A}) in which g is elliptic such that $d(\pi(T), S) \leq 1$.

Proof. First we assume that (1) holds. We can collapse all edges of T that are not in the orbit of e and obtain a free splitting S for (G, \mathcal{A}) where g is elliptic. Hence g is simple. As T collapses to S , we have that $d(\pi(T), S) \leq 1$.

Next we assume that (2) holds. By Lemma 4.3, there is a Grushko tree $T' \in \mathcal{O}$ and a collapse map $T' \rightarrow T$ that restricts to an injection $T'_g \rightarrow T_g$. Thus there is an edge in T' that is not crossed by aT'_g for any $a \in G$ and as in (1) we conclude that T' collapses to a free splitting S for (G, \mathcal{A}) where g is elliptic. Thus g is simple as T' collapses to both T and S , we have that $d(\pi(T), S) \leq 1$. \square

Definition 4.5. Suppose \mathcal{L} is a periodic line collection. We say a Grushko tree $T \in \mathcal{O}$ is *Whitehead reduced for \mathcal{L}* if $\text{Wh}_T(\mathcal{L}, v)$ does not have an admissible cut for any vertex $v \in T$. If $\mathcal{L} = \mathcal{L}_g$ for some non-peripheral element $g \in G$, we say T is *Whitehead reduced for g* .

Lemma 4.6. *Suppose \mathcal{L} is a periodic line collection and that $T \in \mathcal{O}$ is a Grushko tree. Then there exists a Grushko tree $T' \in \mathcal{O}$ with $d(\pi(T), \pi(T')) \leq |\mathcal{L}|_T$ and such that one of the following holds:*

- (1) *T' contains an edge that is not crossed by $\ell_{T'}$ for any line $\ell \in \mathcal{L}$; or*
- (2) *T' is Whitehead reduced for \mathcal{L} and $|\mathcal{L}|_{T'} \leq |\mathcal{L}|_T$.*

Proof. If T is Whitehead reduced or contains an edge that is not crossed by ℓ_T for any line in \mathcal{L} , then set $T' = T$.

Else, there is a vertex $v \in T$ such that $\text{Wh}_T(\mathcal{L}, v)$ has an admissible cut. By Lemma 4.3, there is a Grushko tree $T_1 \in \mathcal{O}$ and a simplicial map $T_1 \rightarrow T$ where $|\mathcal{L}|_{T_1} = |\mathcal{L}|_T$ and for which T_1/G has more edges than T/F (note that T_1 is allowed to have vertices of degree two in the case of a fold). Further, we see that $d(\pi(T_1), \pi(T)) \leq 1$.

If T_1 is Whitehead reduced or contains an edge that is not crossed by ℓ_{T_1} for any line in \mathcal{L} , then set $T' = T_1$. Else, we repeat this process of producing Grushko trees $T_i \in \mathcal{O}$ where $|\mathcal{L}|_{T_i} = |\mathcal{L}|_T$, the number of edges in T_i/G is at least $i + 1$ and $d(\pi(T_i), \pi(T)) \leq i$ as long as T_{i-1} is not Whitehead reduced for \mathcal{L} and every edge in T_{i-1} is crossed by $\ell_{T_{i-1}}$ for some line in \mathcal{L} .

This process must terminate by $L = |\mathcal{L}|_T$. Indeed, the number of orbits of edges in a hypothetical T_L is at least $|\mathcal{L}|_T + 1$ and thus there is an edge $e \subset T_L$ that is not crossed by ℓ_T for any line in \mathcal{L} . \square

Combining Lemmas 4.4 and 4.6, we obtain the following corollary.

Corollary 4.7. *Suppose $g \in G$ is a non-peripheral element that is not simple, $L > 0$ and $T \in \mathcal{O}_L(g)$. Then there exists a Grushko tree $T' \in \mathcal{O}_L(g)$ that is Whitehead reduced for g such that $d(\pi(T), \pi(T')) \leq L$.*

Guirardel–Horbez show that their notion of a Whitehead graph can be used to detect simplicity of a non-peripheral element, generalizing Whitehead’s well-known cut vertex criterion [33].

Proposition 4.8 ([15, Proposition 5.1]). *Suppose $g \in F$ is non-peripheral. Then g is simple if and only if for each Grushko tree $T \in \mathcal{O}$, there is a vertex $v \in T$ such that $\text{Wh}_T(\mathcal{L}_g, v)$ has an admissible cut.*

5. PROOF OF THEOREM 1.1 FOR SIMPLE ELEMENTS

Combining the statements from the previous section, we can prove Theorem 1.1 for simple elements.

Proposition 5.1. *Let (G, \mathcal{A}) be a non-sporadic torsion-free free product. For all $L > 0$, there is a $D_0 > 0$ such that if $g \in G$ is simple, then the diameter of $\pi(\mathcal{O}_L(g)) \subset \mathcal{ZF}$ is at most D_0 .*

Proof. Given L , we set $D_0 = 2L + 3$.

Let $g \in G$ be a simple element and consider a Grushko tree $T_0 \in \mathcal{O}_L(g)$. As g is simple, by Proposition 4.8, the second option of Lemma 4.6 cannot occur. Hence there is a Grushko tree $T \in \mathcal{O}$ where $d(\pi(T_0), \pi(T)) \leq L$ and an edge $e \subset T$ that is not crossed by aT_g for any $g \in G$. By Lemma 4.4, we obtain a free splitting $S_0 \in \mathcal{ZF}$ in which g is elliptic and $d(\pi(T), S_0) \leq 1$. Hence $d(\pi(T_0), S_0) \leq L + 1$.

Given any other Grushko tree $T_1 \in \mathcal{O}_L(g)$, repeating the argument from above, we have that there is a free splitting $S_1 \in \mathcal{ZF}$ in which g is elliptic and $d(\pi(T_1), S_1) \leq L + 1$.

As g is elliptic in both S_0 and S_1 , we have that $d(S_0, S_1) \leq 1$. Therefore, we find:

$$d(\pi(T_0), \pi(T_1)) \leq d(\pi(T_0), S_0) + d(S_0, S_1) + d(S_1, \pi(T_1)) \leq 2L + 3 = D_0. \quad \square$$

Remark 5.2. The above proof also shows that the diameter of the projection of $\mathcal{O}_L(g)$ to the free factor graph FF is also bounded by $2L + 3$. Indeed, one model of the free factor graph (denoted FF_2 by Guirardel–Horbez [16, Definition 2.3]) is defined as the graph with vertex set the \mathcal{Z} -splittings of (G, \mathcal{A}) where two are joined by an edge if they are compatible or if there is a non-peripheral simple element $g \in G$ that is elliptic in both. As the element g in Proposition 5.1 is simple we have that the distance in the free factor graph between the \mathcal{Z} -splittings S_0 and S_1 from above is also at most 1. In the case where $\mathcal{A} = \emptyset$, this was observed by Bestvina–Feighn with an argument that is similar to the one above [2, Lemma 3.3].

6. CONNECTIVITY PROPERTIES OF THE DECOMPOSITION SPACE

To extend Proposition 5.1 to \mathcal{Z} -simple elements that are not simple, we need to analyze the decomposition space $\mathcal{D}(\mathcal{L})$ and its relation to the Whitehead graph $\text{Wh}_T(\mathcal{L}, v)$. Again, let (G, \mathcal{A}) be a fixed non-sporadic torsion-free free product.

6.1. Connectivity. To begin, we observe that the Whitehead graph can detect if the decomposition space is disconnected. Suppose that Y is a direction at p for some point p . Recall that we denote by $\partial_0 Y$ the point p . Further, we denote $\partial Y = \partial T \cap \bar{Y} \subset \partial(G, \mathcal{A})$ where $\bar{Y} \subset \widehat{T}_{\text{obs}}$ is the closure.

Lemma 6.1. *If \mathcal{L} is a periodic line collection and $\text{Wh}_T(\mathcal{L}, v)$ has a type i admissible cut for some Grushko tree $T \in \mathcal{O}$ and vertex $v \in T$, then $\mathcal{D}(\mathcal{L})$ is disconnected.*

Proof. As $\text{Wh}_T(\mathcal{L}, v)$ admits a type i admissible cut, by Lemma 4.3(1), there is a Grushko tree $T' \in \mathcal{O}$ that collapses to T in which some edge $e \subset T'$ is not crossed by $\ell_{T'}$ for any line $\ell \in \mathcal{L}$. Let $p \in T'$ be the midpoint of e , and denote the directions at p by Y and Y' . As these sets are disjoint,

the sets $\partial Y, \partial Y' \subset \partial(G, \mathcal{A})$ are nonempty disjoint clopen sets. Since $\ell_{T'}$ is disjoint from e for all lines $\ell \in \mathcal{L}$ we see that no line $\ell \in \mathcal{L}$ has a point in both ∂Y and $\partial Y'$. Thus the images of the sets ∂Y and $\partial Y'$ remain disjoint in $\mathcal{D}(\mathcal{L})$ and also remain clopen. Hence $\mathcal{D}(\mathcal{L})$ is disconnected. \square

We seek to prove a converse to this statement and hence to give a characterization of when the decomposition space $\mathcal{D}(\mathcal{L})$ is connected.

Lemma 6.2. *Let \mathcal{L} be a periodic line collection. Suppose that $T \in \mathcal{O}$ is a Grushko tree that is Whitehead reduced for \mathcal{L} . Consider an edge e in T and let x be the midpoint of e . If $Y \subset T - \{x\}$ is a direction at x , then the subset $q(\partial Y) \subset \mathcal{D}(\mathcal{L})$ is connected.*

Proof. Suppose there are open sets $A, B \in \mathcal{D}(\mathcal{L})$ such that $q(\partial Y) \subseteq A \cup B$ and $A \cap B \cap q(\partial Y) = \emptyset$. We will prove the lemma by showing that either $A \cap q(\partial Y) = \emptyset$ or $B \cap q(\partial Y) = \emptyset$.

We observe that the sets:

$$\begin{aligned} Y_A &= \partial Y \cap q^{-1}(A) = \partial Y \cap q^{-1}(\mathcal{D}(\mathcal{L}) - B), \\ Y_B &= \partial Y \cap q^{-1}(B) = \partial Y \cap q^{-1}(\mathcal{D}(\mathcal{L}) - A) \end{aligned}$$

are both compact clopen sets in $\partial(G, \mathcal{A})$. As the intersection $A \cap B \cap q(\partial Y)$ is empty, there is no line in \mathcal{L} with one point in Y_A and the other point in Y_B .

As Y_A and Y_B are compact clopen sets, there are finitely many edges in T , with midpoints $P = \{p_1, \dots, p_m\}$, such that for a point $\xi \in \partial Y$, membership in Y_A or Y_B is determined by the parity of the intersection of the set P with the ray r representing ξ based at x . In this case, we say that the set P determines the partition $Y_A \sqcup Y_B$ of ∂Y . Observe that there are many possible sets which can determine the same partition of ∂Y .

If P is non-empty, we will describe a new set of points P' that determines the same partition $Y_A \sqcup Y_B$ of ∂Y such that P' has strictly fewer points than P . Repeating this process a finite number of times, we see that the partition $Y_A \sqcup Y_B$ can be determined by the empty set. Hence either Y_A or Y_B is empty, proving that $q(\partial Y)$ is connected.

To this end, fix a point $p \in P$ that maximizes the distance to x , let v be the vertex incident to the edge containing p that is closest to x and let e' be the edge incident to v in the direction at v that contains x . Let q' be the midpoint of e' and enumerate the midpoints of the edges incident to v other than e' by $q_1 = p, \dots, q_i, \dots$. Note the possibility that $q' = x$. There is an integer M such that, after possibly reordering the q_i 's, we have that $q_i \in P$ if $1 \leq i \leq M$ and $q_i \notin P$ otherwise. Without loss of generality, we can say that a geodesic ray originating at x that contains q_i for some $1 \leq i \leq M$ represents a point in Y_A while a geodesic ray originating at x that contains q_i for some $i > M$ represents a point in Y_B . (This uses the fact the p maximizes the distance to x .) In particular, there is no line $\ell \in \mathcal{L}$ such that ℓ_T contains that points q_i and q_j where $1 \leq i \leq M$ and $j > M$. Let $Y' \subset T - \{v\}$ be the direction that contains q' and let $Y_i \subset T - \{v\}$ be the direction that contains q_i for $i \geq 1$. As vertices in the Whitehead graph $\text{Wh}_T(\mathcal{L}, v)$, we see that there are no edges from $[Y_i]$ to $[Y_j]$ where $1 \leq i \leq M$ and $j > M$.

See Figure 3 for the set-up.

First suppose that $\text{Stab}_T(v) = \{1\}$ and let δ be the valence of v . If $M < \delta - 1$, then we see that $\text{Wh}_T(\mathcal{L}, v)$ has an admissible cut, contrary to our hypotheses. Indeed, either $\text{Wh}_T(\mathcal{L}, v)$ is disconnected or $[Y']$ is a cut vertex. Hence we have that $M = \delta - 1$ and so each of the q_i 's belong to P . In this case, let P' be the symmetric difference between P and the set $\{q', q_1, \dots, q_M\} - \{x\}$. In other words, we replace q_1, \dots, q_M by q' if $q' \notin P$ and $q' \neq x$, and remove q', q_1, \dots, q_M otherwise.

We are now left with the case that $\#|\text{Stab}_T(v)| = \infty$. We will argue that this case cannot occur. Let Y_{i_1}, \dots, Y_{i_m} be a sequence of directions at v where:

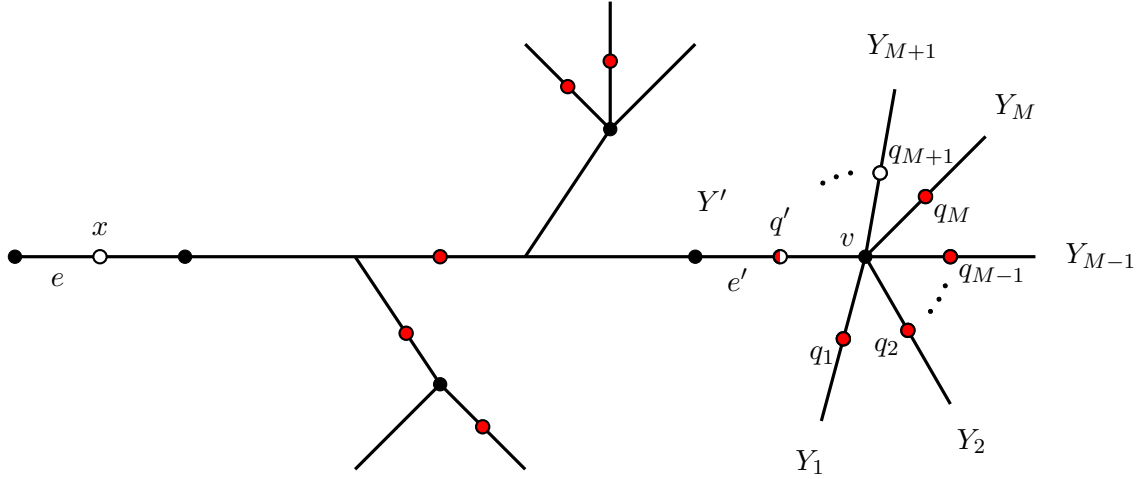


FIGURE 3. The set-up in the proof of Lemma 6.2. Vertices of T are filled in black, points that below in P are filled in red and other midpoints of edges are filled in white. The point q' may or not belong to P .

- (1) there is a line $\ell_j \in \mathcal{L}$ such that $(\ell_j)_T$ contains q_{i_j} and $q_{i_{j+1}}$ for $1 \leq j < m$, and
- (2) $Y_{i_1} = Y_1$ and $Y_{i_m} = aY_1$ for some $a \in \text{Stab}_T(v)$.

Such a sequence gives rise to a closed path in $\text{Wh}_T(\mathcal{L}, v)$ based at $[Y_1]$. Moreover, the product of the labels along this path in $\text{Wh}_T(\mathcal{L}, v)$ is a .

Claim. The element a is trivial.

Proof of Claim. We observe that $1 \leq i_j \leq M$ for all $1 \leq j \leq m$ as there is no line $\ell \in \mathcal{L}$ such that ℓ_T contains that points q_i and q_j where $1 \leq i \leq M$ and $j > M$. In particular, $aq_1 \in P$. We can concatenate the original sequence with $aY_1, aY_{i_2}, \dots, aY_{i_m} = a^2Y_1$ to observe that $a^2q_1 \in P$ also. Repeating in this manner we have that $a^nq_1 \in P$ for all $n \geq 0$. As P is a finite set, we must have that a is trivial as claimed. \square

This implies that $\text{Wh}_T(\mathcal{L}, v)$ has an admissible cut, contrary to our hypothesis. Indeed, let U be the connected component of the subgraph spanned by the vertices $[Y_1], \dots, [Y_M]$ that contains the vertex $[Y_1]$. By the claim we have that $\text{Mon}(U) = \{1\}$. If U is a connected component of $\text{Wh}_T(\mathcal{L}, v)$, then there is a type i admissible cut. Else, let U' be the subgraph spanned by U and $[Y']$. Arguing as in the claim, we can also see that $\text{Mon}(U') = \{1\}$. In this case it is now clear that there is a type ii admissible cut. \square

We can now state our converse to Lemma 6.1.

Proposition 6.3. *Let \mathcal{L} be a periodic line collection. The following are equivalent.*

- (1) *There exists a Grushko tree $T \in \mathcal{O}$ that is Whitehead reduced for \mathcal{L} .*
- (2) *The decomposition space $\mathcal{D}(\mathcal{L})$ is connected.*

Proof. Suppose there exists a Grushko tree $T \in \mathcal{O}$ that is Whitehead reduced for \mathcal{L} . Fix an edge $e \in T$, let x be the midpoint of e and denote the directions at x by Y and Y' . By Lemma 6.2, both $q(\partial Y)$ and $q(\partial Y')$ are connected. Further, there is a geodesic that crosses the edge e as otherwise the Whitehead graph based at either of the vertices incident to e has an isolated vertex and hence

an admissible cut. Thus $q(\partial Y) \cap q(\partial Y') \neq \emptyset$ and therefore $\mathcal{D}_{\mathcal{L}} = q(\partial Y) \cup q(\partial Y')$ is connected. Hence (1) implies (2).

Next suppose that there does not exist a Grushko tree $T \in \mathcal{O}$ that is Whitehead reduced for \mathcal{L} . By Lemma 4.6, there is a Grushko tree $T' \in \mathcal{O}$ that contains an edge that is not crossed by any geodesic in \mathcal{L} . As the proof of Lemma 6.1 shows, this implies that $\mathcal{D}(\mathcal{L})$ is disconnected. Hence (2) implies (1). \square

We obtain the following corollary of Proposition 6.3 using Corollary 4.7.

Corollary 6.4. *Suppose $g \in G$ is a non-peripheral element. Then g is not simple if and only if $\mathcal{D}(\mathcal{L}_g)$ is connected.*

6.2. Cut sets in decomposition spaces. In this section we introduce the terminology needed to talk about cut sets in decomposition spaces and record how \mathcal{Z} -splittings give rise to cut sets.

Let D be a Peano continuum, i.e., a compact, connected, locally connected, metrizable space. A finite set $P \subset D$ is a *cut set* if $D - P$ is disconnected but $D - P'$ is connected for any proper subset $P' \subset P$. We use the terms *cut point*, respectively *cut pair*, when $\#|P| = 1$, or when $\#|P| = 2$. A *local cut point* is a point $x \in D$ which is either a cut point or where $D - \{x\}$ is connected and has more than one end. In the latter case, the *valence* of x is the number of ends of $D - \{x\}$, denoted $\text{val}(x)$. A cut pair $\{x, y\} \subset D$ is *exact* if $\text{val}(x) = \text{val}(y) = \#|\pi_0(D - \{x, y\})|$. In other words, each component of $D - \{x, y\}$ has two ends, one that limits to x and one that limits to y . It is known that the valence of a point in an exact cut pair must be finite [22, Section 2.2]. The *valence* of an exact cut pair is the valence of either of its two points. We will use the terms *bivalent*, or *multivalent* respectively, to mean that the valence is equal to 2, or greater than or equal to 3.

An exact cut pair $\{y_0, y_1\} \subset D$ *separates* an exact cut pair $\{x_0, x_1\} \subset D$, if x_0 and x_1 lie in different components of $D - \{y_0, y_1\}$. It can be readily checked that separating is a symmetric notion among exact cut pairs [7, Lemma 2.3]. An exact cut pair $\{x_0, x_1\} \subset D$ is *inseparable* if it is not separated by another exact cut pair. As the notion of separating is symmetric, if $\{x_0, x_1\}, \{y_0, y_1\} \subset D$ are exact pairs and $\{x_0, x_1\}$ is inseparable, then we have that both x_0, x_1 belong to the same component of $D - \{y_0, y_1\}$ and y_0, y_1 belong to the same component of $D - \{x_0, x_1\}$.

Recently, Dasgupta–Hruska proved that the Bowditch boundary of a countable relatively hyperbolic group is a Peano continuum whenever it is connected [11, Theorem 1.1]. Hence the above discussion, by Lemma 3.2, applies to a connected decomposition space $\mathcal{D}(\mathcal{L})$. A *loxodromic cut pair* in a decomposition space $\mathcal{D}(\mathcal{L})$ is a cut pair of the form $P = \{q(a^\infty), q(a^{-\infty})\}$ where a is a non-peripheral element of $(G, \mathcal{A} \cup \mathcal{N}_{\mathcal{L}})$. We remark that loxodromic cut pairs are always exact [22, Lemma 4.1].

Lemma 6.5. *Suppose that $g \in G$ is a \mathcal{Z} -simple element that is not simple. Then $\mathcal{D}(\mathcal{L}_g)$ contains a loxodromic cut pair.*

Proof. Suppose that g is \mathcal{Z} -simple but not simple and let S be a \mathcal{Z} -splitting in which g is elliptic. As g is not simple, every edge in S has nontrivial stabilizer. Fix an element $a \in G$ that generates the stabilizer of an edge in S .

As g is not simple, we have that $\mathcal{D}(\mathcal{L}_g)$ is connected by Corollary 6.4. Let $\mathcal{L} = \mathcal{L}_g \cup \mathcal{L}_a$. Then as $\mathcal{D}(\mathcal{L})$ the quotient of $\mathcal{D}(\mathcal{L}_g)$, it is also connected and by Lemma 3.2, it is homeomorphic to the Bowditch boundary of the relatively hyperbolic group $(G, \mathcal{A} \cup \mathcal{N}_{\mathcal{L}})$. As $(G, \mathcal{A} \cup \mathcal{N}_{\mathcal{L}})$ splits over $\langle a \rangle$, which is a peripheral subgroup of $(G, \mathcal{A} \cup \mathcal{N}_{\mathcal{L}})$, by work of Dasgupta–Hruska [11, Theorem 1.1] we find that $\mathcal{D}(\mathcal{L})$ contains a cut point which is the fixed point of the subgroup $\langle a \rangle$. The pre-image of

this cut point in $\mathcal{D}(\mathcal{L}_g)$ is the two point set $\{q(a^\infty), q(a^{-\infty})\}$. This pair is thus a loxodromic cut pair. \square

Remark 6.6. When $\mathcal{A} = \emptyset$, so that G is a free group, it is easy to see the conclusion of Lemma 6.5 directly. Indeed, for simplicity, assume that there is a \mathcal{Z} -splitting corresponding to an amalgamated free product decomposition $G = A *_{\langle a \rangle} B$ where $g \in A$. As g is not simple, we have that g is not commensurable to a . There is a 2-complex M obtained from two graphs Γ_A and Γ_B where $\pi_1(\Gamma_A) \cong A$ and $\pi_1(\Gamma_B) \cong B$ by attaching a cylinder $S^1 \times [0, 1]$ where $S^1 \times \{0\}$ is glued to a loop in Γ_A representing $a \in A$ and $S^1 \times \{1\}$ is glued to a loop in Γ_B representing $a \in B$. The universal cover \widetilde{M} is quasi-isometric to G and so we can identify $\partial\widetilde{M}$ with ∂G . Then $S^1 \times \{1/2\} \subset M$ lifts to an embedded line in $L \subset \widetilde{M}$ whose endpoints corresponds to $a^\infty, a^{-\infty} \in \partial G$. By construction, ∂G decomposes into two sets C_0 and C_1 , corresponding to the Gromov boundaries of the two components of $\widetilde{M} - L$, where $C_0 \cap C_1 = \{a^\infty, a^{-\infty}\}$. Let $C'_0 = C_0 - \{a^\infty, a^{-\infty}\}$ and $C'_1 = C_1 - \{a^\infty, a^{-\infty}\}$. For each $h \in G$, the pair $\{hg^\infty, hg^{-\infty}\}$ belongs either to C'_0 or to C'_1 . Hence $q(C'_0)$ and $q(C'_1)$ are open disjoint sets and $\mathcal{D}(\mathcal{L}_g) - \{q(a^\infty), q(a^{-\infty})\} = q(C'_0) \cup q(C'_1)$ showing that $\{q(a^\infty), q(a^{-\infty})\}$ is a cut pair.

7. MODELING THE DECOMPOSITION SPACE VIA WHITEHEAD GRAPHS

We require a notion of the Whitehead graph based on the complement of a locally finite subtree $X \subset T$ in order to model the decomposition space. In the setting where $\mathcal{A} = \emptyset$, this idea was fully developed and investigated by Cashen–Macura [8] (see the references within as well for early work in this direction). In that setting, the definition using a locally finite subtree $X \subset T$ is exactly as it is for a sole vertex $v \in T$: the vertices of the Whitehead graph are the directions of $T - X$ and two are joined by an edge for each line $\ell \in \mathcal{L}$ such that ℓ_T meets both directions. There is a way to build up such graphs by “splicing” together Whitehead graphs based at vertices. In the general free product setting the naive approach does not work as sometimes the $\text{Stab}_T(\partial_0 Y)$ -orbit of a component $Y \subset T - X$ meets X , which complicates this picture.

After first defining our notion of a Whitehead graph in this general setting in Section 7.1, we relate it to cut sets in the decomposition space in Lemma 7.4 and Section 7.4.

Let (G, \mathcal{A}) be a fixed non-sporadic torsion-free free product.

7.1. Whitehead graph relative to a locally finite subtree. We begin by producing a structure modeling the relations between the lines and directions at a vertex with non-trivial stabilizer.

Suppose $T \in \mathcal{O}$ is a Grushko tree, \mathcal{L} is a periodic line collection, and $v \in T$ is a vertex with non-trivial stabilizer. For simplicity, we will assume that T is Whitehead reduced for \mathcal{L} as this is the only case where we are concerned with in the sequel. We define a graph $T_v(\mathcal{L})$ whose vertices correspond to directions at v and two such directions $Y, Y' \subset T - \{v\}$ are connected by an edge for line $\ell \in \mathcal{L}$ such that ℓ_T meets both Y and Y' . Note that $T_v(\mathcal{L})$ is locally finite as \mathcal{L} has finitely many G -orbits and hence each edge $e \subset T$ meets only finitely many ℓ_T .

Suppose $\widetilde{U} \subseteq T_v(\mathcal{L})$ is a connected subgraph and $G_{\widetilde{U}}$ is the subgroup of $\text{Stab}_v(T)$ that leaves \widetilde{U} invariant. Then $G_{\widetilde{U}}$ acts freely on \widetilde{U} and the quotient $\widetilde{U}/G_{\widetilde{U}}$ is isomorphic to a connected component U of the Whitehead graph $\text{Wh}_T(\mathcal{L}, v)$. Moreover, $G_{\widetilde{U}}$ is in the conjugacy class $\text{Mon}(U)$. In particular, as $\text{Wh}_T(\mathcal{L}, v)$ does not have a type i admissible cut and each of the peripheral subgroups are torsion-free, each component of $T_v(\mathcal{L})$ contains infinitely many vertices.

Consider a locally finite subtree $X \subset T$. Note that we allow X to have vertices of degree one that are not vertices of T . We define $T_v(\mathcal{L}) - X$ to be the subgraph of $T_v(\mathcal{L})$ spanned by the directions at v that are disjoint from X . That is, we remove from $T_v(\mathcal{L})$ the directions at v that meet X

and all of these incident edges. We will use the structure of this graph to create a set, based on the components of $T_v(\mathcal{L}) - X$, that will appear as vertices in the Whitehead graph $\text{Wh}_T(\mathcal{L}, X)$. Specifically, by $V_X(\mathcal{L}, v)$ we denote the union of the vertices of $T_v(\mathcal{L}) - X$ that belong to a finite component of $T_v(\mathcal{L}) - X$ with an additional element denoted \hat{v} . The element \hat{v} represents each direction at v that belongs to an infinite component of $T_v(\mathcal{L}) - X$. As X is a locally finite tree, $T_v(\mathcal{L})$ is locally finite, and each component of $T_v(\mathcal{L})$ contains infinitely many vertices, the set $V_X(\mathcal{L}, v)$ is a finite set.

The Whitehead graph based at X is defined as follows. The vertex set of $\text{Wh}_T(\mathcal{L}, X)$ is the union of $V_X(\mathcal{L}, v)$ over all vertices v of T contained in X with infinite stabilizer, together with the set of directions $Y \subset T - X$ where $\text{Stab}_T(\partial_0 Y)$ is trivial. Two distinct vertices of $\text{Wh}_T(\mathcal{L}, X)$ are connected by an edge for each line $\ell \in \mathcal{L}$ such that ℓ_T has non-trivial intersection with directions represented by the two vertices. As for the version of the Whitehead graph defined in Section 4, the vertex representing a direction $Y \subset T - X$ is denoted by $[Y]$. The vertex $[Y]$ in $\text{Wh}_T(\mathcal{L}, X)$ is called *trivial* if $\text{Stab}_T(\partial_0 Y) = \{1\}$ and is called *infinite* if $\#|\text{Stab}_T(\partial_0 Y)| = \infty$. We note that this dichotomy is well-defined as $[Y] = [Y']$ implies that $\partial_0 Y = \partial_0 Y'$.

When X is a single vertex v where $\text{Stab}_T(v) = \{1\}$ this notion of Whitehead graph agrees with the definition of Guirardel–Horbez given above, which is the classical notion of a Whitehead graph. When $\#|\text{Stab}_T(v)| = \infty$ the above notion of Whitehead graph will always consist of a single vertex and no edges. We will never make use of this notion. Hence, in the sequel when we are referring to a Whitehead graph relative to a single vertex, i.e., $\text{Wh}_T(\mathcal{L}, v)$, we are always using the Guirardel–Horbez definition.

Example 7.1. We consider the free product (G, \mathcal{A}) , Grushko tree $T \in \mathcal{O}$ and element $g \in G$ as in Example 4.1. We will also make use of the notation from that example. The graph $T_v(\mathcal{L}_g)$ is the disjoint union of four lines as shown in Figure 4.

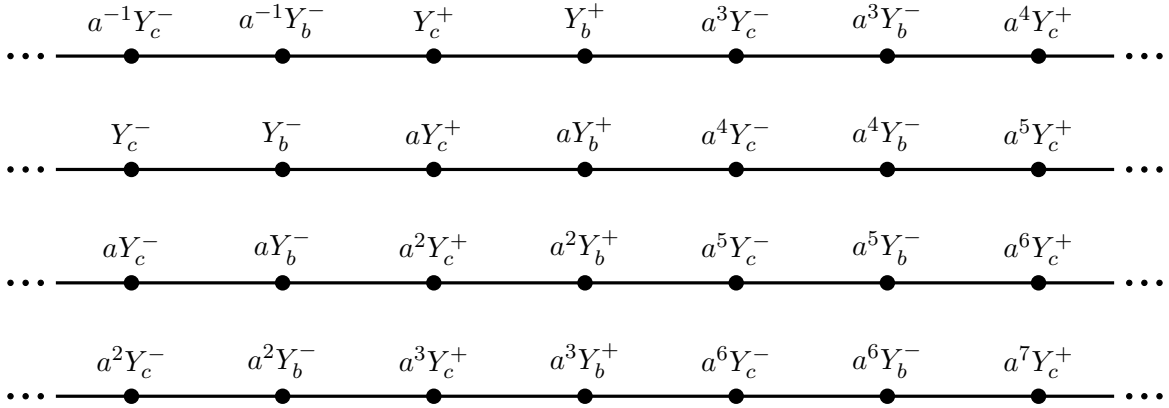


FIGURE 4. The graph $T_v(\mathcal{L}_g)$ in Example 7.1.

Subdivide the edge e_1 into $e_1^+e_1^-$ and likewise subdivide e_2 into $e_2^+e_2^-$. Let X be the subtree of T that is the union of the edges $a^{-1}b^{-1}e_1^-$, e_1 , $bc^{-1}e_2^-$ and $ba^{-4}b^{-1}e_1^-$. There are two vertices of T that belong to X , namely v and bv . For simplicity, we denote bv by w . There are only two vertices of $T_v(\mathcal{L}_g)$ that meet X , specifically $a^{-1}Y_b^-$ and Y_b^+ . Referring to Figure 4, we see that removing these vertices from $T_v(\mathcal{L}_g)$ results in a graph with six connected components: three that are lines, two that are rays and one that only consists of the single vertex Y_c^+ . Thus $V_X(\mathcal{L}_g, v)$ consists of two elements: Y_c^+ and \hat{v} .

Translation by b defines a bijection between the directions at v and the directions at w . We use the letter “ Z ” to denote the corresponding directions at w , e.g., $Z_b^+ = bY_b^+$, $aZ_c^- = b(aY_c^-)$, etc. We see that X meets the directions Z_b^- , Z_c^- , and $a^{-4}Z_b^-$. Using the Figure for $T_v(\mathcal{L}_g)$ in Figure 4, we find that $V_X(\mathcal{L}_g, w)$ consists of three elements: $a^{-3}Z_b^+$, $a^{-3}Z_c^+$, and \hat{w} .

There are three directions $Y \subset T - X$ where $\text{Stab}_T(\partial_0 Y)$ is trivial. The direction contained in $a^{-1}Y_b^-$ is denoted $a^{-1}W_b^-$ (note: $a^{-1}Y_b^- = a^{-1}b^{-1}e_1^- \cup a^{-1}W_b^-$). Likewise we denote the direction contained in Z_c^- by W_c^- and the direction contained in $a^{-4}Z_b^-$ by $a^{-4}W_b^-$. This set-up is shown in Figure 5. The Whitehead graph $\text{Wh}_T(\mathcal{L}_g, X)$ is shown in Figure 6.

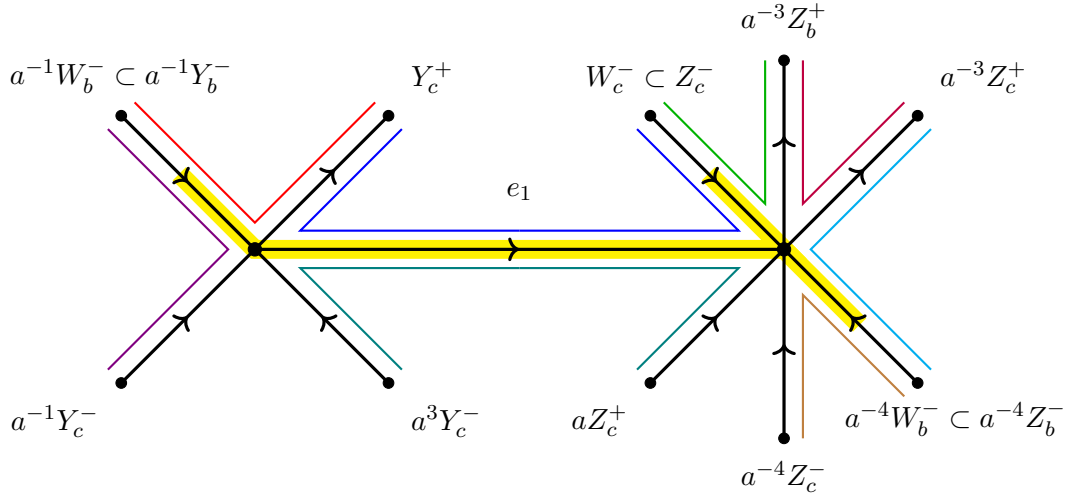


FIGURE 5. The set-up for the subtree X , shown in yellow, and some translates of T_g in Example 7.1.

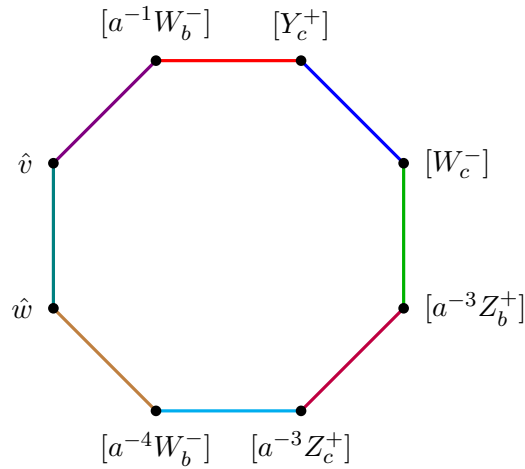


FIGURE 6. The Whitehead graph $\text{Wh}_T(\mathcal{L}_g, X)$ in Example 7.1.

The following two lemmas will be used to help analyze the structure of $\text{Wh}_T(\mathcal{L}, X)$. An *embedded line* in a graph Γ is an injective function $\alpha: \mathbb{R} \rightarrow \Gamma$ which is simplicial using the standard simplicial structure on \mathbb{R} where the vertex set is equal to \mathbb{Z} .

Lemma 7.2. *Let \mathcal{L} be a periodic line collection. Suppose that $T \in \mathcal{O}$ is a Grushko tree and $\text{Wh}_T(\mathcal{L}, v)$ that is Whitehead reduced for \mathcal{L} . For any vertex $v \in V_\infty(T)$ and vertex Y of $T_v(\mathcal{L})$, there is an embedded line $\alpha_Y: \mathbb{R} \rightarrow T_v(\mathcal{L})$ such that $\alpha_Y(0) = Y$.*

Proof. Fix a vertex $Y \in T_v(\mathcal{L})$, let \tilde{U} be the component of $T_v(\mathcal{L})$ that contains Y , and let $G_{\tilde{U}}$ be the subgroup of $\text{Stab}_T(v)$ that leaves \tilde{U} invariant. As above, we identify $\tilde{U}/G_{\tilde{U}}$ with a component U in $\text{Wh}_T(\mathcal{L}, v)$ where $G_{\tilde{U}}$ is in the conjugacy class $\text{Mon}(U)$. In particular, since $\text{Wh}_T(\mathcal{L}, v)$ does not have a type i admissible cut, we have that $G_{\tilde{U}}$ is nontrivial. As $\text{Stab}_T(v)$ is torsion-free, every nontrivial element in $G_{\tilde{U}}$ has infinite order.

First, suppose that $[Y]$ is a cut vertex of U . Write U as the union $U_1 \cup U_2$ where U_1 and U_2 are connected, and $[Y] = U_1 \cap U_2$. As $\text{Wh}_T(\mathcal{L}, v)$ does not have a type ii admissible cut both of the subgraphs U_1 and U_2 have non-trivial monodromy. Thus, there are embedded cycles $\gamma_1 \subseteq U_1$ and $\gamma_2 \subseteq U_2$, where $\text{Mon}(\gamma_1)$ and $\text{Mon}(\gamma_2)$ are non-trivial, i.e., the products of the labels along both γ_1 and γ_2 are non-trivial. Let β be the embedded path (possibly trivial) in U that connects γ_1 to γ_2 . Necessarily the path β contains $[Y]$. The subgraph $\gamma_1 \cup \beta \cup \gamma_2$ lifts to an infinite tree in \tilde{U} that contains Y but no valence one vertices. See the graph on the left in Figure 7. Hence there is an embedded line that contains Y .

Next suppose that $[Y]$ is not a cut vertex of U . Let U_0 be the biconnected component of U that contains $[Y]$. As $\text{Wh}_T(\mathcal{L}, v)$ does not have a type ii admissible cut, the subgraph U_0 has non-trivial monodromy.

Claim. The vertex $[Y]$ lies on an embedded cycle $\gamma \subseteq U_0$ where $\text{Mon}(\gamma)$ is non-trivial.

Proof of Claim. Fix some embedded cycle $\gamma_0 \subseteq U_0$ where $\text{Mon}(\gamma_0)$ is non-trivial. Assuming that $[Y]$ does not lie on γ_0 , there is an embedded path β that contains $[Y]$ and only intersects γ_0 in its endpoints. This follows as U_0 is biconnected. The endpoints of β decompose γ_0 into two subpaths γ'_0 and γ''_0 . Then $[Y]$ lies on the two cycles $\beta \cdot \gamma'_0$ and $\beta \cdot \gamma''_0$. As γ_0 has non-trivial monodromy, at least one of these cycles has non-trivial monodromy. See the graph on the right in Figure 7. \square

The cycle γ lifts to an embedded line in \tilde{U} that contains $[Y]$. \square

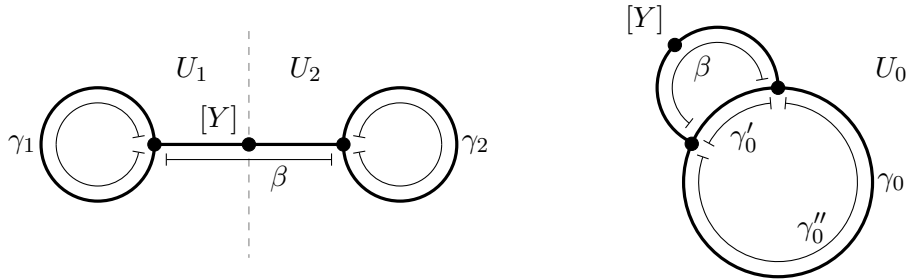


FIGURE 7. The two cases of Lemma 7.2.

Let v be a vertex in T . A *finite star about v* is the minimal connected subset $Z \subset T$ that contains a finite set of points that are midpoints of edges incident on v .

Lemma 7.3. *Let \mathcal{L} be a periodic line collection. Suppose that $T \in \mathcal{O}$ is a Grushko tree that is Whitehead reduced for \mathcal{L} and $v \in V_\infty(T)$. Let $Z \subset T$ be a finite star about v . Then $\text{Wh}_T(\mathcal{L}, Z)$ is connected and no vertex is a cut vertex other than possibly \hat{v} .*

Proof. Enumerate the trivial directions at Z by Y_1, \dots, Y_m . Each of these directions is contained in a unique direction at v . Abusing notation, we will use Y_1, \dots, Y_m to denote these corresponding directions at v . These are exactly the vertices deleted when forming $T_v(\mathcal{L}) - Z$. Notice that the vertex set of $\text{Wh}_T(\mathcal{L}, Z)$ is the union of $\{[Y_1], \dots, [Y_m]\}$ with $V_Z(\mathcal{L}, v)$.

Fix a vertex $Y \in T_v(\mathcal{L})$ and let $\alpha_Y: \mathbb{R} \rightarrow T_v(\mathcal{L})$ be an embedded line such that $\alpha_Y(0) = Y$. Such a line exists by Lemma 7.2. There is some $M \geq 0$ such that $\alpha_Y(j) \notin \{Y_1, \dots, Y_m\}$ for all $j \geq M$. In particular, the vertex $[\alpha_Y(M)]$ is \hat{v} . As $\alpha_Y(j-1)$ and $\alpha_Y(j)$ are adjacent in $T_v(\mathcal{L})$ for all $j \in \mathbb{Z}$, we have that the corresponding vertices $[\alpha_Y(j-1)]$ and $[\alpha_Y(j)]$ in $\text{Wh}_T(\mathcal{L}, Z)$ are also adjacent if they are distinct. This shows that there is an edge path from $[Y]$ to \hat{v} . Hence $\text{Wh}_T(\mathcal{L}, Z)$ is connected.

Suppose that Y is a vertex of $T_v(\mathcal{L})$ and $[Y] \neq \hat{v}$ in $\text{Wh}_T(\mathcal{L}, Z)$. Thus Y is the only direction at Z that corresponds to $[Y]$. If $[Y]$ is a cut vertex of $\text{Wh}_T(\mathcal{L}, Z)$, then we can write $\text{Wh}_T(\mathcal{L}, Z)$ as the union of two graphs U and V where $U \cap V = [Y]$. Without loss of generality, we can assume that $\hat{v} \notin U$ and furthermore that U is connected. Take a vertex $[Y'] \in U$ other than $[Y]$ and let $\alpha_{Y'}: \mathbb{R} \rightarrow T_v(\mathcal{L})$ be an embedded line such that $\alpha_{Y'}(0) = Y'$. Again, such a line exists by Lemma 7.2. As above there is an $M \geq 0$ such that $\alpha_{Y'}(j) \notin \{Y_1, \dots, Y_m\}$ for $j \leq -M$ or $j \geq M$. This gives two paths in $\text{Wh}_T(\mathcal{L}, Z)$ from $[Y']$ to \hat{v} but only one of them can contain $[Y]$ as $\alpha_{Y'}$ is an embedded line. This is a contradiction, hence $[Y]$ is not a cut vertex. \square

7.2. Relating the decomposition space to Whitehead graphs. The definition of the Whitehead graph $\text{Wh}_T(\mathcal{L}, X)$ is justified by the next lemma.

Lemma 7.4. *Let \mathcal{L} be a periodic line collection and let $P \subseteq \mathcal{D}(\mathcal{L})$ be a finite subset where $q^{-1}(P) \cap V_\infty(G, \mathcal{A})$ is empty and where $q^{-1}(P)$ contains more than one point. Suppose that $T \in \mathcal{O}$ is a Grushko tree that is Whitehead reduced for \mathcal{L} and let $X \subset T$ be the convex hull of $q^{-1}(P)$. Then there is a bijection between the connected components of $\text{Wh}_T(\mathcal{L}, X)$ and the connected components of $\mathcal{D}(\mathcal{L}) - P$.*

Proof. Suppose Y is a direction at X and let $\partial_0 Y = v$. If $\text{Stab}_T(v) = \{1\}$ or if Y belongs to a finite component of $T_v(\mathcal{L}) - X$, then $[Y] \neq \hat{v}$ and moreover, the subset $q(\partial Y) \subset \mathcal{D}(\mathcal{L}) - P$ is connected by Lemma 6.2.

Next, let $v \in X$ be a vertex with $\text{Stab}_T(v) \neq \{1\}$ and enumerate the directions at v that belong to an infinite component of $T_v(\mathcal{L}) - X$ by $\{Y_i\}$. As above, we have that each subset $q(\partial Y_i) \subset \mathcal{D}(\mathcal{L}) - P$ is connected by Lemma 6.2.

Claim. The set $\{q(v)\} \cup \bigcup_i q(\partial Y_i)$ is contained in a connected component of $\mathcal{D}(\mathcal{L}) - P$.

Proof of Claim. Observe that if, as vertices of $T_v(\mathcal{L}) - X$, the directions Y_i and Y_j are adjacent, then $q(\partial Y_i) \cap q(\partial Y_j) \neq \emptyset$. Indeed, as these directions are adjacent in $T_v(\mathcal{L}) - X$ there is a line $\ell \in \mathcal{L}$ such that ℓ_T meets both Y_i and Y_j and whose endpoints do not lie in P . The endpoints of this line lie in ∂Y_i and ∂Y_j respectively and are mapped by q to the same point in $\mathcal{D}(\mathcal{L})$. Continuing in this manner, it follows that if Y_i and Y_j lie in the same component of $T_v(\mathcal{L}) - X$ then $q(\partial Y_i)$ and $q(\partial Y_j)$ lie in the same component of $\mathcal{D}(\mathcal{L}) - P$.

Let $\{Y_{i_j}\}$ be an enumeration of the directions at v that belong to a fixed infinite component of $T_v(\mathcal{L}) - X$. To complete the proof of the claim we will show that $q(v)$ belongs to the same component of $\mathcal{D}(\mathcal{L}) - P$ as $\bigcup_j q(\partial Y_{i_j})$. To see this, fix points $y_j \in \partial Y_{i_j}$. As $y_j \rightarrow v$ in the observers topology, we have that $q(y_j) \rightarrow q(v)$ since q is continuous and therefore $q(v)$ lies in the same component as $\bigcup_j q(\partial Y_{i_j})$. \square

Now consider a connected component $U \subseteq \text{Wh}_T(\mathcal{L}, X)$. Let $\{Y_i\}$ be the complete collection of directions at X that correspond to vertices in U and let $\{v_j\}$ be the collection of vertices in X where $\text{Stab}_T(v) \neq \{1\}$ and such that $v_j = \partial_0 Y_i$ for some Y_i . Then the above observations imply that the set:

$$\bigcup_j q(v_j) \cup \bigcup_i q(\partial Y_i) \quad (7.1)$$

is connected and open in $\mathcal{D}(\mathcal{L}) - P$. As the same is true for any connected component of $\text{Wh}_T(\mathcal{L}, X)$ and as $\partial(G, \mathcal{A}) - q^{-1}(P)$ is the union of $q(\partial Y)$ over all directions $Y \subset T - X$ together with the union of $q(v)$ over all vertices $v \in V_\infty(T)$, we see that the set in (7.1) is also closed and therefore a connected component of $\mathcal{D}(\mathcal{L}) - P$. \square

7.3. Splicing Whitehead graphs. There is a splicing procedure for building $\text{Wh}_T(\mathcal{L}, X)$ as described by Manning [28] and used by Cashen–Macura [8] in the setting where $\mathcal{A} = \emptyset$. This procedure allows one to study $\text{Wh}_T(\mathcal{L}, X)$ by considering the Whitehead graphs relative to the vertices in X . We will describe a modification of splicing for the free-product setting. In this setting, we consider a locally finite subtree $X \subset T$ as being constructed by gluing together finite stars.

To this end let $X \subset T$ be a finite subtree and let $Z \subset X$ be a finite star about some vertex v in X such that $X' = X - Z$ is connected. We will describe how to construct $\text{Wh}_T(\mathcal{L}, X)$ using $\text{Wh}_T(\mathcal{L}, X')$ and $\text{Wh}_T(\mathcal{L}, Z)$. Let $Y \subset T - X'$ be the direction that contains Z and let $Y' \subset T - Z$ be the direction that contains X' . We observe that $\partial_0 Y = \partial_0 Y'$ is the midpoint of an edge, which we denote by e . As the stabilizer of this point is trivial, we see that Y is the only direction at X' that represents the vertex $[Y]$ in $\text{Wh}_T(\mathcal{L}, X')$. Likewise Y' is the only direction at Z that represents $[Y']$ in $\text{Wh}_T(\mathcal{L}, Z)$. Let \mathcal{L}_e be the collection of lines in \mathcal{L} such that ℓ_T contains e . Each line in \mathcal{L}_e corresponds to a unique edge incident to $[Y]$ in $\text{Wh}_T(\mathcal{L}, X')$ and also to a unique edge incident to $[Y']$ in $\text{Wh}_T(\mathcal{L}, Z)$. This defines a bijection between the set of edges incident to $[Y]$ and the set of edges incident to $[Y']$. Now we remove the vertex $[Y]$ from $\text{Wh}_T(\mathcal{L}, X')$ and the vertex $[Y']$ from $\text{Wh}_T(\mathcal{L}, Z)$, maintaining the edges with “loose ends.” We will denote these graphs with loose ends by $\text{Wh}_T(\mathcal{L}, X') - \{[Y]\}$ and $\text{Wh}_T(\mathcal{L}, Z) - \{[Y']\}$ respectively. We now *splice* together these loose ends using the bijection coming from \mathcal{L}_e . The resulting graph is $\text{Wh}_T(\mathcal{L}, X)$.

Example 7.5. We consider the free product (G, \mathcal{A}) , Grushko tree $T \in \mathcal{O}$ and element $g \in G$ as in Examples 4.1 and 7.1. We will make use of the notation from those examples. Let p be the midpoint of the edge e_1 . We can decompose X along this point and write X as the union of two finite stars Z and Z' where $v \in Z$, $w \in Z'$ and $Z \cap Z' = \{p\}$. The Whitehead graphs $\text{Wh}_T(\mathcal{L}_g, Z)$ and $\text{Wh}_T(\mathcal{L}_g, Z')$ are shown in Figure 8. We obtain $\text{Wh}_T(\mathcal{L}_g, X)$ by removing the vertex $[Y_b^+]$ from $\text{Wh}_T(\mathcal{L}_g, Z)$ and vertex $[Z_b^-]$ and splicing the like colored loose ends together.

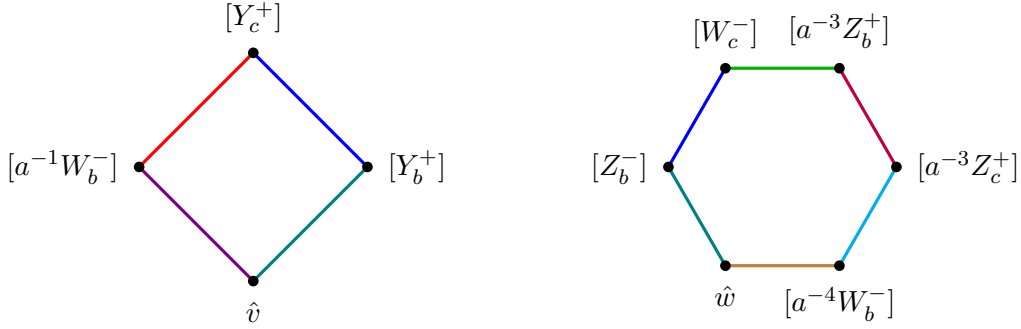
7.4. Identifying cut sets in $\mathcal{D}(\mathcal{L})$ via Whitehead graphs. Suppose that $T \in \mathcal{O}$ is Whitehead reduced for \mathcal{L} and v is a vertex of T . Using the Whitehead graph $\text{Wh}_T(\mathcal{L}, v)$, we can identify two types of cut sets in $\mathcal{D}(\mathcal{L})$. These will be used in the next two sections to break the decomposition space up along cut points and to characterize quadratic elements in G .

Definition 7.6. If either:

- (i) $\text{Wh}_T(\mathcal{L}, v)$ is disconnected, or
- (ii) $\text{Wh}_T(\mathcal{L}, v)$ is connected and $\text{Mon}(\text{Wh}_T(\mathcal{L}, v))$ is not equal to $\text{Stab}_T(v)$,

then $q(v) \in \mathcal{D}(\mathcal{L})$ is called a *peripheral cut point*.

Definition 7.7. Fix an edge e that is incident to v and let $\mathcal{L}_e \subset \mathcal{L}$ be the collection of lines in \mathcal{L} such that ℓ_T contains e for each $\ell \in \mathcal{L}_e$. Then the set $q(\mathcal{L}_e) \subset \mathcal{D}(\mathcal{L})$ is called an *edge cut set*.

FIGURE 8. The Whitehead graphs $\text{Wh}_T(\mathcal{L}_g, Z)$ and $\text{Wh}_T(\mathcal{L}_g, Z')$ in Example 7.5.

Lemma 7.8. *Suppose that \mathcal{L} is a periodic line collection and that $\mathcal{D}(\mathcal{L})$ is connected. A peripheral cut point is a cut point. Moreover, a point $x \in \mathcal{D}(\mathcal{L})$ is a cut point only if either x is a peripheral cut point or $x = q(\ell)$ for some $\ell \in \mathcal{L}$.*

Proof. First we observe that peripheral cut points are indeed cut points of $\mathcal{D}(\mathcal{L})$. To this end, let $v \in T$ be a vertex such that $q(v)$ is a peripheral cut point. Notice that either of the conditions in Definition 7.6 imply that $T_v(\mathcal{L})$ is not connected. Let $\tilde{U} \subset T_v(\mathcal{L})$ be a connected component. Then the set:

$$C_{\tilde{U}} = \bigcup_{Y \in V(\tilde{U})} q(\partial(Y)) \subset \mathcal{D}(\mathcal{L})$$

is an open set. Indeed, the union of the sets $\partial(Y) \subset \partial(G, \mathcal{A})$ over the directions at v determined by \tilde{U} is an open saturated set. Further it is connected by Lemma 6.2 and since for $Y, Y' \in V(\tilde{U})$ that are adjacent in \tilde{U} , we have that $q(\partial(Y)) \cap q(\partial(Y'))$ is non-empty. As $\mathcal{D}(\mathcal{L}) - \{q(v)\}$ is the union of $C_{\tilde{U}'}$ over all components $\tilde{U}' \subset T_v(\mathcal{L})$ and $C_{\tilde{U}} \cap C_{\tilde{U}'} = \emptyset$ for $\tilde{U} \neq \tilde{U}'$, we see that $C_{\tilde{U}}$ is closed as well. This shows that $q(v)$ is indeed a cut point.

Next, suppose that $x \in \mathcal{D}(\mathcal{L})$ is a cut point. By Lemma 3.2, the decomposition space $\mathcal{D}(\mathcal{L})$ is the boundary of the relatively hyperbolic group $(G, \mathcal{A} \cup \mathcal{N}_{\mathcal{L}})$. Dasgupta–Hruska proved that every cut point of the boundary of a relatively hyperbolic group is fixed by a subgroup conjugate into $\mathcal{A} \cup \mathcal{N}_{\mathcal{L}}$ [11, Theorem 1.1]. Hence either $x = q(v)$ for some $v \in V_{\infty}(G, \mathcal{A})$ or $x = q(\ell)$ for some $\ell \in \mathcal{L}$. We claim in the former case that $q(v)$ is a peripheral cut point. Indeed, by Proposition 6.3, there exists a Grushko tree $T \in \mathcal{O}$ that is Whitehead reduced for \mathcal{L} . Suppose that $q(v)$ is not a peripheral cut point. Then $\text{Wh}_T(\mathcal{L}, v)$ is connected and $\text{Mon}(\text{Wh}_T(\mathcal{L}, v)) = \text{Stab}_T(v)$. Hence $T_v(\mathcal{L})$ is connected and the above argument shows that $\mathcal{D}(\mathcal{L}) - \{q(v)\}$ is connected as well. Indeed, in the notation above we have $C_{\tilde{U}} = \mathcal{D}(\mathcal{L}) - \{q(v)\}$. This contradicts the assumption that $q(v)$ is a cut point. Hence we have that $q(v)$ is a peripheral cut point. \square

We need the following notation. Suppose that $X \subset T$ is a locally finite tree and \mathcal{L}' is a subset of lines in \mathcal{L} . Then $\text{Wh}_T(\mathcal{L}, X) - \mathcal{L}'$ is the subgraph of $\text{Wh}_T(\mathcal{L}, X)$ obtained by removing every edge that corresponds to a line in \mathcal{L}' .

Lemma 7.9. *Let \mathcal{L} be a periodic line collection. If $\mathcal{D}(\mathcal{L})$ is connected and every cut point is a peripheral cut point, then an edge cut set is a cut set.*

Proof. The statement and proof are similar to statements by Cashen–Macura [8, Lemma 4.20 & Proposition 4.21].

Fix an edge $e \subset T$ and let $\mathcal{L}_e \subset \mathcal{L}$ be the collection of lines in \mathcal{L} as given by Definition 7.7. Let $Y \subset T - \{v\}$ the direction at v that meets e .

Let

$$X = \bigcup_{\ell \in \mathcal{L}_e} \ell_T \subset T.$$

By Lemma 7.4 we have that there is a bijection between the components of $\mathcal{D}(\mathcal{L}) - q(\mathcal{L}_e)$ and $\text{Wh}_T(\mathcal{L}, X)$.

Let $Z_0 \subset X$ be the minimal closed subset such that X equals the union of Z_0 together with the images of the $2|\mathcal{L}_e|$ geodesic rays $\rho_\ell^\pm: [0, \infty] \rightarrow \widehat{T}$ with disjoint interiors where $\rho_\ell^\pm(\infty) = \ell_T^\pm$. This set Z_0 is what Cashen–Macura call the *core* [8, Section 4.6]. We need to modify it slightly for our purposes. We set

$$Z = Z_0 \cup \bigcup_{\ell \in \mathcal{L}_e} \rho_\ell^+([0, 1/2]) \cup \bigcup_{\ell \in \mathcal{L}_e} \rho_\ell^-([0, 1/2]).$$

In other words, we add on the initial half-edges of each of the geodesic rays ρ_ℓ^\pm to the set Z_0 . See Figure 9. Note that Z is the finite union of finite stars. We remark that each line $\ell \in \mathcal{L}_e$ determines an edge in $\text{Wh}_T(\mathcal{L}, Z)$ that is incident to two trivial vertices.

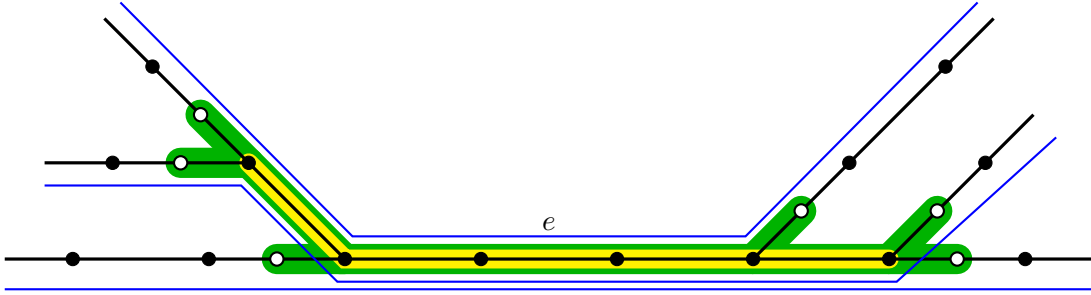


FIGURE 9. The lines in \mathcal{L}_e (blue) together with a schematic of the subsets Z_0 (yellow) and Z (green) from Lemma 7.9. Vertices of T are shown in black and midpoints of edge are shown in white.

Claim 1. There is a bijection between the connected components of $\text{Wh}_T(\mathcal{L}, X)$ and $\text{Wh}_T(\mathcal{L}, Z) - \mathcal{L}_e$.

Proof of Claim 1. For each geodesic ray ρ_ℓ^\pm , we let $R_\ell^\pm = \rho_\ell^\pm([1/2, \infty)) \subset T$ and we let Y_ℓ^\pm be the direction at $\rho_\ell^\pm(1/2)$ that meets Z . Observe that $\text{Wh}_T(\mathcal{L}, R_\ell^\pm) - \{[Y_\ell^\pm]\}$ is connected. Indeed, as $q(\ell)$ is not a cut point, by Lemma 7.4, we have that $\text{Wh}_T(\mathcal{L}, \ell_T)$ is connected. Now $\text{Wh}_T(\mathcal{L}, R_\ell^+, \mathcal{L}) - \{[Y_\ell^+]\}$ is an infinite end of $\text{Wh}_T(\mathcal{L}, \ell_T)$ and as the stabilizer of ℓ_T acts cocompactly on $\text{Wh}_T(\mathcal{L}, \ell_T)$ we see that $\text{Wh}_T(\mathcal{L}, R_\ell^+) - \{[Y_\ell^+]\}$ must be connected as well. The same holds for $\text{Wh}_T(\mathcal{L}, R_\ell^-) - \{[Y_\ell^-]\}$.

Now let \overline{Y}_ℓ^\pm be the direction at $\rho_\ell^\pm(1/2)$ that does not meet Z . Then $\text{Wh}_T(\mathcal{L}, X)$ is obtained from $\text{Wh}_T(\mathcal{L}, Z) - \mathcal{L}_e$ by removing the vertices $[\overline{Y}_\ell^\pm]$ and splicing this graph with loose ends together with the graphs with loose ends $\text{Wh}_T(\mathcal{L}, R_\ell^\pm) - \{[Y_\ell^\pm]\}$. As the latter graphs are connected, there is a clear bijection between the connected components of $\text{Wh}_T(\mathcal{L}, X)$ and $\text{Wh}_T(\mathcal{L}, Z) - \mathcal{L}_e$. \square

Next, we will show how we can trim the set Z down to the set which is essentially the edge e . Specifically, for each of the lines $\ell \in \mathcal{L}_e$, let e_ℓ^\pm be the two edges contained in ℓ that share a single vertex with e and so that e_ℓ^\pm separates e from ℓ_T^\pm . We note the possibility that some of the edges

e_ℓ^\pm might be the same. Subdivide each e_ℓ^\pm into $e_{\ell,0}^\pm e_{\ell,1}^\pm$ where $e_{\ell,0}^\pm$ shares a vertex with e . Finally, we set:

$$\hat{e} = e \cup \bigcup_{\ell \in \mathcal{L}_e} e_{\ell,0}^+ \cup \bigcup_{\ell \in \mathcal{L}_e} e_{\ell,0}^-.$$

As for $Z \subset T$, each line $\ell \in \mathcal{L}_e$ determines an edge in $\text{Wh}_T(\mathcal{L}, \hat{e})$ that is incident to two trivial vertices.

The following claim, and its proof, are similar to the discussion of the *pruned core* [8, Section 4.6].

Claim 2. There is a bijection between the connected components of $\text{Wh}_T(\mathcal{L}, Z) - \mathcal{L}_e$ and $\text{Wh}_T(\mathcal{L}, \hat{e}) - \mathcal{L}_e$.

Proof of Claim 2. Let $Z' \subset Z$ be a finite star about some vertex $v \in Z$ such that $Z - Z'$ is connected and contains e . Set $\hat{Z} = Z - Z'$. We will show that there is a bijection between the connected components of $\text{Wh}_T(\mathcal{L}, Z) - \mathcal{L}_e$ and $\text{Wh}_T(\mathcal{L}, \hat{Z}) - \mathcal{L}_e$. The claim then follows by repeating this process until we have trimmed the set Z down to \hat{e} .

We consider the Whitehead graph $\text{Wh}_T(\mathcal{L}, Z')$. Let Y be the direction at Z' that contains e and let Y' be the direction at \hat{Z} that contains Z' . We observe that $\text{Wh}_T(\mathcal{L}, Z) - \mathcal{L}_e$ is obtained from $\text{Wh}_T(\mathcal{L}, \hat{Z}) - \mathcal{L}_e$ by removing the vertex $[Y']$ and splicing this to the graph with loose ends obtained from $\text{Wh}_T(Z', \mathcal{L}) - \mathcal{L}_e$ by removing $[Y]$.

The Whitehead graph $\text{Wh}_T(\mathcal{L}, Z')$ is connected and $[Y]$ is not a cut vertex of this graph. Indeed, if $\text{Stab}_T(v) = \{1\}$, then this follows as $\text{Wh}_T(\mathcal{L}, Z')$ is isomorphic to $\text{Wh}_T(\mathcal{L}, v)$ and so $\text{Wh}_T(\mathcal{L}, Z')$ does not have an admissible cut. Else, if $\#|\text{Stab}_T(v)| = \infty$ this follows by Lemma 7.3.

As each of the edges in $\text{Wh}_T(\mathcal{L}, Z')$ corresponding to one of the lines in \mathcal{L}_e is incident to $[Y]$, we have that the graph with loose ends $\text{Wh}_T(\mathcal{L}, Z') - \mathcal{L}_e$ obtained by removing $[Y]$ is also connected. Hence as in Claim 1, we are removing a vertex and splicing on a connected graph with loose ends and there is a clear bijection between the connected components. \square

The lemma now follows as $\text{Wh}_T(\mathcal{L}, \hat{e}) - \mathcal{L}_e$ contains exactly two components (this follows from another application of Lemma 7.3) and adding any of the edges corresponding to one of the lines $\ell \in \mathcal{L}_e$ connects these two components. \square

8. DECOMPOSING $\mathcal{D}(\mathcal{L}_g)$ ALONG CUT POINTS

Now that we have a model for the decomposition space, we want to work with it to identify \mathcal{Z} -splittings of our free product (G, \mathcal{A}) where a given \mathcal{Z} -simple element is elliptic. To this end, it will become necessary to be able to promote the existence of a loxodromic cut pair in the decomposition space to the existence of an inseparable loxodromic cut pair, as the latter actually corresponds to a \mathcal{Z} -splitting of the free product. We start with a proposition (Proposition 8.1) that allows for such a promotion when the decomposition space is not homeomorphic to the circle nor contains cut points. We deal with the case that the decomposition space is homeomorphic to a circle in Section 9. The remainder of this section is dealing with the case of cut points in the decomposition space. We will apply recent work of Dasgupta–Hruska [11] that constructs a tree that models the structure of cut points in the boundary of a relatively hyperbolic group. The key features of this tree appear in Lemma 8.3. We will see that we can apply Proposition 8.1 to a certain vertex stabilizer in this tree. The ultimate statement regarding the existence of inseparable loxodromic cut pairs appears in Section 9 as Corollary 9.5 as we must further analyze the case that the aforementioned vertex stabilizer also produces a decomposition space that is homeomorphic to a circle.

Let (G, \mathcal{A}) be a fixed non-sporadic torsion-free free product.

Proposition 8.1. *Let \mathcal{L} be a periodic line collection and suppose that $\mathcal{D}(\mathcal{L})$ is connected and contains a loxodromic cut pair. Then one of the following holds:*

- (1) $\mathcal{D}(\mathcal{L})$ is homeomorphic to a circle.
- (2) $\mathcal{D}(\mathcal{L})$ contains a cut point.
- (3) $\mathcal{D}(\mathcal{L})$ contains an inseparable loxodromic cut pair.

Proof. Assume that \mathcal{L} is a periodic line collection so that $\mathcal{D}(\mathcal{L})$ is connected and contains a loxodromic cut pair. Further suppose that $\mathcal{D}(\mathcal{L})$ is not homeomorphic to the circle and also that $\mathcal{D}(\mathcal{L})$ does not contain a cut point.

Let $\{q(a^\infty), q(a^{-\infty})\}$ be a loxodromic cut pair in $\mathcal{D}(\mathcal{L})$. It is well-known that multivalent exact cut pairs are inseparable, see [7, Lemma 2.3] or [22, Lemma 2.6] for instance. Hence, we can assume that $\{q(a^\infty), q(a^{-\infty})\}$ is bivalent. In the setting of boundaries of relatively hyperbolic groups, Hallmark–Hruska [22, Proposition 4.6] characterize bivalent local cut points when the boundary is connected without cut points and not homeomorphic to the circle. Their result applies to $\mathcal{D}(\mathcal{L})$ by Lemma 3.2. Their characterization implies that a bivalent exact cut pair is either an inseparable loxodromic cut pair or it is part of a *necklace*. The definition of a necklace is not necessary, we only require the fact that each necklace contains a *jump*, which itself is an inseparable loxodromic cut pair (cf. [21, Lemma 4.19]). \square

In the case that $\mathcal{A} = \emptyset$, the above proposition appears as Corollary 4.10 in work of Cashen [7]. Again, the proof given by Cashen applies mutatis mutandis to the free product setting. We chose the above proof for brevity.

Let $g \in G$ be a non-peripheral element that is not simple. By Corollary 6.4 we have that $\mathcal{D}(\mathcal{L}_g)$ is connected. Further, if we assume that g is \mathcal{Z} -simple, then $\mathcal{D}(\mathcal{L}_g)$ contains a loxodromic cut pair by Lemma 6.5 and so Proposition 8.1 applies. As mentioned above, in the next section we will analyze the case that $\mathcal{D}(\mathcal{L}_g)$ is homeomorphic to a circle, or more generally an inverse limit of a tree of circles. In the remainder of this section we will see how we can decompose the decomposition space along cut points.

Dasgupta–Hruska have described a tree \mathcal{T}_{cut} that models the structure of cut points in $\mathcal{D}(\mathcal{L}_g)$ [11, Theorem 1.2]. The tree has a natural bipartite structure on \mathcal{T}_{cut} where each vertex is either:

- (1) a nontrivial maximal subcontinuum of $\mathcal{D}(\mathcal{L}_g)$ that is not separated by a cut point in $\mathcal{D}(\mathcal{L}_g)$,
or
- (2) a cut point in $\mathcal{D}(\mathcal{L}_g)$.

The set of vertices of the first type is denoted $V_0(\mathcal{T}_{\text{cut}})$; the set of vertices of the second type is denoted $V_1(\mathcal{T}_{\text{cut}})$. A vertex $B \in V_0(\mathcal{T}_{\text{cut}})$ is adjacent to a vertex $x \in V_1(\mathcal{T}_{\text{cut}})$ if $x \in B$. The group G acts on \mathcal{T}_{cut} in the obvious way. If $\mathcal{D}(\mathcal{L}_g)$ does not have any cut points, then \mathcal{T}_{cut} consists of a single vertex corresponding to $\mathcal{D}(\mathcal{L}_g)$.

Lemma 8.2. *Let $g \in G$ be a non-peripheral element that is not simple. Then the following hold.*

- (1) *There is a unique vertex in \mathcal{T}_{cut} that is fixed by g . Moreover, this vertex belongs to $V_0(\mathcal{T}_{\text{cut}})$.*
- (2) *Any edge stabilizer in \mathcal{T}_{cut} is peripheral.*

Proof. Let $B \subseteq \mathcal{D}(\mathcal{L}_g)$ be a maximal subcontinuum of $\mathcal{D}(\mathcal{L}_g)$ that is not separated by a cut point in $\mathcal{D}(\mathcal{L}_g)$ and that contains $q(g^\infty)$.

If g does not fix B as a vertex in $V_0(\mathcal{T}_{\text{cut}})$, then $q(g^\infty)$ is a cut point and hence a vertex in $V_1(\mathcal{T}_{\text{cut}})$. Let S be the (G, \mathcal{A}) tree obtained by collapsing all edges of \mathcal{T}_{cut} other than those in the orbit of the edge connecting $q(g^\infty)$ to B . The result is a \mathcal{Z} -splitting of (G, \mathcal{A}) where the stabilizer of some edge in S is a subgroup of $\langle g \rangle$. As we assumed that g is not simple, this is a contradiction. Indeed, if this edge stabilizer is trivial, then S witnesses g as simple. Else, if this edge stabilizer is

nontrivial, then Lemma 2.2 implies that some power of g , and hence also g , is simple. Therefore g fixes B .

If g fixes a vertex other than B , then g fixes a cut point in $\mathcal{D}(\mathcal{L}_g)$, as is seen using the bipartite structure of the tree \mathcal{T}_{cut} . As g is non-peripheral, by Lemma 7.8, this can only happen if $q(g^\infty)$ is a cut point. As above, this is a contradiction. This shows (1).

By the bipartite structure of \mathcal{T}_{cut} , any edge stabilizer must fix a cut point in $\mathcal{D}(\mathcal{L}_g)$. By Lemma 7.8, any cut point is either a peripheral cut point or equal to $q(ag^\infty)$ for some $a \in G$. Once again, the above shows that the latter possibility cannot occur, hence any cut point is a peripheral cut point and therefore stabilized by a peripheral subgroup. Hence any edge stabilizer is also a peripheral subgroup. This shows (2). \square

For a vertex $B \in V_0(\mathcal{T}_{\text{cut}})$ we let $G_B = \text{Stab}_{\mathcal{T}_{\text{cut}}}(B)$. By \mathcal{N}_g^B we denote the G_B -conjugacy class of $N_{g'}$ where $g' \in B$ is conjugate to g . If no such conjugate exists we set $\mathcal{N}_g^B = \emptyset$. By Lemma 8.2(1) this is well-defined.

In order to state the following lemma, which will allow us to promote the existence of a loxodromic cut pair to the existence of an inseparable loxodromic cut pair even in the setting of cut points with one additional hypothesis, we need the following notation introduced by Guirardel–Levitt [19]. Again, consider a vertex $B \in V_0(\mathcal{T}_{\text{cut}})$. By Inc_B we denote the set of G_B -conjugacy classes of stabilizers of the edges in \mathcal{T}_{cut} incident to B . By $\mathcal{A}|_{G_B}$ we denote the set of G_B -conjugacy classes of subgroups hA_ih^{-1} for $i = 1, \dots, k$ that fix B but not any incident edge. Lastly, we set $\mathcal{P}_B = \text{Inc}_B \cup \mathcal{A}|_{G_B}$. By Lemma 8.2(2), we have that any subgroup of G_B whose conjugacy class is in \mathcal{P}_B is necessarily peripheral.

Lemma 8.3. *Let $g \in G$ be a non-peripheral element that is \mathcal{Z} -simple but not simple and let $B \in V_0(\mathcal{T}_{\text{cut}})$ be the unique vertex fixed by g . Then the following hold.*

- (1) *The pair (G_B, \mathcal{P}_B) is a torsion-free free product and g is non-peripheral with respect to \mathcal{P}_B .*
- (2) *The element g is \mathcal{Z} -simple and not simple with respect to \mathcal{P}_B .*
- (3) *The vertex set $V_0(\mathcal{T}_{\text{cut}})$ has a single orbit.*
- (4) *For the periodic line pattern $\mathcal{L}_g^B = \{\{ag^\infty, ag^{-\infty}\} \mid a \in G_B\}$ on the free product (G_B, \mathcal{P}_B) , the decomposition space $\mathcal{D}_{(G_B, \mathcal{P}_B)}(\mathcal{L}_g^B)$ is homeomorphic to the boundary of the relatively hyperbolic group $(G_B, \mathcal{P}_B \cup \mathcal{N}_g^B)$.*
- (5) *Either the decomposition space $\mathcal{D}_{(G_B, \mathcal{P}_B)}(\mathcal{L}_g)$ is homeomorphic to a circle, or it contains an inseparable loxodromic cut pair.*
- (6) *Any inseparable loxodromic cut pair in $\mathcal{D}_{(G_B, \mathcal{P}_B)}(\mathcal{L}_g)$ is an inseparable loxodromic cut pair in $\mathcal{D}(\mathcal{L}_g)$.*

Proof. First, Dasgupta–Hruska observe that $G_{B'}$ is finitely generated for any $B' \in V_0(\mathcal{T}_{\text{cut}})$ [11, Proposition 2.13]. This implies that (G_B, \mathcal{P}_B) is a torsion-free free product. Indeed, take any Grushko tree $T \in \mathcal{O}$ and consider $T_B \subseteq T$ the minimal subtree for the action of G_B on T . Suppose that H is a subgroup of G_B whose G_B -conjugacy class is in \mathcal{P}_B . Then H is peripheral and hence has a unique fixed point in T_B . Moreover, any non-trivial vertex stabilizer in T_B is of the form $H_B = G_B \cap hA_ih^{-1}$ for some $i = 1, \dots, k$ and $h \in G$. If H_B is a proper subgroup of hA_ih^{-1} then we claim that H_B fixes an edge incident to B . Indeed, let $v \in V_\infty(T) \subseteq \partial(G, \mathcal{A})$ be the vertex fixed by hA_ih^{-1} . As H_B is a proper subgroup of hA_ih^{-1} , then hA_ih^{-1} does not fix B . Thus $q(v)$ is a cut point of $\mathcal{D}(\mathcal{L}_g)$, and hence a vertex in $V_1(\mathcal{T}_{\text{cut}})$. Then clearly the edge incident on B and $q(v)$ is stabilized by H_B , completing our claim. Therefore the conjugacy class of any vertex stabilizer in T_B is in \mathcal{P}_B . As edge stabilizers in T_B are trivial and G_B is finitely generated, this shows that (G_B, \mathcal{P}_B) is a torsion-free free product.

As g is non-peripheral with respect to (G, \mathcal{A}) and all subgroups represented in \mathcal{P}_B are peripheral with respect to \mathcal{A} , we see that g is non-peripheral with respect to \mathcal{P}_B as well. This shows (1).

Let S be a \mathcal{Z} -splitting of (G, \mathcal{A}) where g is elliptic and let $S_B \subseteq S$ be the minimal subtree with respect to the action of G_B on S . As any subgroup of G_B whose conjugacy class is in \mathcal{P}_B is necessary peripheral with respect to \mathcal{A} , we see that S_B is a \mathcal{Z} -splitting of (G_B, \mathcal{P}_B) where g is elliptic. This shows that g is \mathcal{Z} -simple with respect to \mathcal{P}_B .

Suppose that g is simple with respect to \mathcal{P}_B and let S be a free splitting of (G_B, \mathcal{P}_B) in which g is elliptic. Then we can blow-up \mathcal{T}_{cut} by equivariantly replacing the vertex B by a copy of the tree S . Indeed, the stabilizers of edges incident on B fix a unique vertex in S and so this is well-defined. The result is a (G, \mathcal{A}) tree which we denote S' . Fix a single edge e in $S \subseteq S'$. Then the result of collapsing all edges of S' outside of the orbit of e is a free splitting of (G, \mathcal{A}) in which g is elliptic. This would imply that g is simple with respect to \mathcal{A} . Therefore, as g is not simple with respect to \mathcal{A} , it is also not simple with respect to \mathcal{P}_B . This shows (2).

Next, suppose that there is a vertex $B' \in V_0(\mathcal{T}_{\text{cut}})$ that is not in the orbit of B . As no conjugate of g fixes B' , we have that $\mathcal{N}_g^{B'} = \emptyset$. As in (1), we have that $(G_{B'}, \mathcal{P}_{B'})$ is a torsion-free free product. As in (2), we have that $(G_{B'}, \mathcal{P}_{B'})$ does not have any free splitting. Therefore $(G_{B'}, \mathcal{P}_{B'})$ must be either $(\{1\}, \emptyset)$, $(G_{B'}, \{[G_{B'}]\})$, or (\mathbb{Z}, \emptyset) . As the action of G on \mathcal{T}_{cut} is minimal, we must have that $\text{Inc}_{B'} = \emptyset$ in all cases. This implies that any edge incident to B' has trivial stabilizer, which can be used to produce a free splitting of (G, \mathcal{A}) , in which g is elliptic. This contradicts the assumption that g is not simple with respect to \mathcal{A} . Hence there is only one orbit of vertices in $V_0(\mathcal{T}_{\text{cut}})$. This shows (3).

We note that (4) follows immediately from Lemma 3.2 and the fact shown by Dasgupta–Hruska prove that $(G_B, \mathcal{P}_B \cup \mathcal{N}_g^B)$ is relatively hyperbolic with boundary homeomorphic to B [11, Theorem 1.2(2)].

For (5), we apply Proposition 8.1 to the decomposition space $\mathcal{D}_{(G_B, \mathcal{P}_B)}(\mathcal{L}_g^B)$ and note that B , which is homeomorphic to $\mathcal{D}_{(G_B, \mathcal{P}_B)}(\mathcal{L}_g^B)$, is connected and does not have a cut point.

To prove (6), we start with $\{x_0, x_1\}$ that is an inseparable loxodromic cut pair in $\mathcal{D}_{(G_B, \mathcal{P}_B)}(\mathcal{L}_g^B)$. As this cut pair is loxodromic, we that $x_0 = q_{(G_B, \mathcal{P}_B)}(a^\infty)$ and $x_1 = q_{(G_B, \mathcal{P}_B)}(a^\infty)$ for some non-peripheral element $a \in G_B$. The element a is non-peripheral with respect to \mathcal{A} as described above in (2) and hence neither x_0 nor x_1 are cut points of $\mathcal{D}(\mathcal{L}_g)$. It is now clear that $\{x_0, x_1\} \subset \mathcal{D}(\mathcal{L}_g)$ is a loxodromic cut pair. Since any cut pair in $\mathcal{D}(\mathcal{L}_g)$ is contained in some $B' \in V_0(\mathcal{T}_{\text{cut}})$, (for instance, see [22, Lemma 2.5]) we have that $\{x_0, x_1\}$ is also an inseparable loxodromic cut pair in $\mathcal{D}(\mathcal{L}_g)$. \square

9. QUADRATIC ELEMENTS IN FREE PRODUCTS

In this section we investigate the case where $\mathcal{D}(\mathcal{L}_g)$, or $\mathcal{D}_{(G_B, \mathcal{P}_B)}(\mathcal{L}_g^B)$ in the case of cut points, is homeomorphic to a circle. The main result in this section is Proposition 9.4 which characterizes such elements $g \in G$. To begin, we recall the definition of a quadratic element in the setting of free products due to Guirardel–Horbez [16, Definition 2.14]. In the setting of free groups, such an element corresponds to the boundary of a connected surface with a single boundary component.

Let (G, \mathcal{A}) be a fixed non-sporadic torsion-free free product.

Definition 9.1. Let g be a non-peripheral element of (G, \mathcal{A}) that is not simple. We say g is *quadratic* if there is a (possibly trivial) (G, \mathcal{A}) -tree Q such that:

- (1) For some vertex v in Q , there is a compact, connected 2-orbifold Σ such that $\text{Stab}_Q(v) \cong \pi_1(\Sigma)$.
- (2) Every edge of Q has a translate that is incident to v .

- (3) Every incident edge group and every peripheral subgroup in $\text{Stab}_Q(v)$ is conjugate into a boundary or a conical subgroup of $\pi_1(\Sigma)$.
- (4) The stabilizer of any edge is peripheral and the stabilizer of any vertex that is not in the orbit of v is peripheral.
- (5) The element g is the generator for a boundary component of Σ .

As remarked by Guirardel–Horbez [16], a subgroup of $\pi_1(\Sigma)$ corresponding to a boundary component of Σ other than the one generated by g is necessarily peripheral as g is not simple. As we are assuming that G is torsion-free, we may assume that Σ in the definition is a surface and hence there are no conical subgroups.

Example 9.2. Let G be the free group of rank five with a basis $\{a, b, c, d, e\}$ and consider the free factor system $\mathcal{A} = \{[\langle a, b \rangle], [\langle e \rangle]\}$. Fix a non-trivial element $x \in \langle a, b \rangle$ and we consider the free product with amalgamation decomposition of G as $\langle a, b \rangle *_{\langle x \rangle} \langle x, c, d, e \rangle$. There is an identification of $\langle x, c, d, e \rangle$ with $\pi_1(\Sigma)$ where Σ is a torus with three boundary components, one corresponding to x , another to e and the last to $g = d^{-1}xcdc^{-1}e$. The Bass-Serre tree Q associated to the splitting shows that g is quadratic. The graph of spaces corresponding to this splitting is shown in Figure 10.

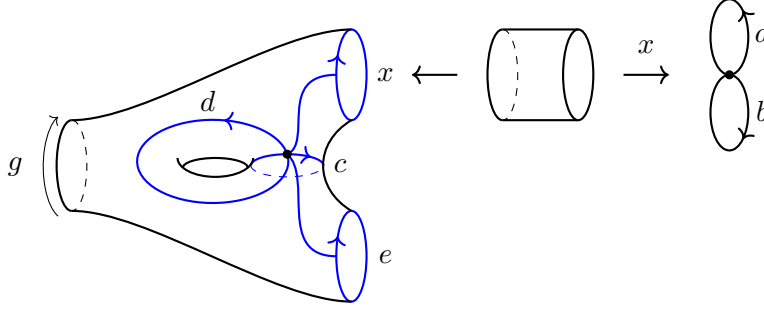


FIGURE 10. The quadratic element in (G, \mathcal{A}) from Example 9.2.

We will now describe the decomposition space $\mathcal{D}(\mathcal{L}_g)$. Let v be the vertex of Q fixed by $\langle c, d, e, x \rangle$, let w be the vertex of Q fixed by $\langle a, b \rangle$ and let ε be the edge incident to v and w . Set $G_v = \{c, d, e, x\}$ and $\mathcal{P} = \{[\langle e \rangle], [\langle x \rangle], [\langle g \rangle]\}$. Then as a relatively hyperbolic group, the boundary of (G_v, \mathcal{P}) is homeomorphic to a circle, which we denote by C_v . We replace the vertex v in Q by C_v and for each $h \in G_v$, we attach the end of the edge $h\varepsilon$ incident to v to the unique point fixed by $h x h^{-1}$ in C_v . Repeat this step in an equivariant way at each vertex in the orbit of v replacing the vertex v' by the circle $C_{v'}$ and attaching the edges as before. Now collapse every translate of ε in the resulting space. This new space, which we will denote Q_C , can be described as the union of the circles $C_{v'}$ over vertices v' in the orbit of v where $C_{v'} \cap C_{v''}$ consists of a single point if the simplicial distance between v' and v'' in Q is 2, and is empty otherwise. The decomposition space $\mathcal{D}(\mathcal{L}_g)$ is the union of $Q_C \cup \partial_\infty Q$. The topology on this space is similar to the observers topology. Specifically, suppose that (x_n) is a sequence of points in Q_C and let v_n be a corresponding sequence of vertices in Q where $x_n \in C_{v_n}$ (note this sequence (v_n) is not unique). By passing to a subsequence of (v_n) , we can assume that $v_n \rightarrow v_\infty$ in \hat{T}_{obs} . There are three cases that we describe now.

- (1) If $v_\infty \in \partial_\infty Q$, then $x_n \rightarrow v_\infty$.
- (2) Suppose that v_n is eventually constant. Thus $x_n \in C_{v_\infty}$ for large enough n . As C_{v_∞} is compact, we can pass to a subsequence to get $x_n \rightarrow x_\infty$ where $x_\infty \in C_{v_\infty}$.
- (3) Otherwise, for each n , there is a unique point $x'_n \in C_{v_\infty}$ such that Q_C decomposes into two connected subsets Q_0 and Q_1 where $Q_0 \cap Q_1 = \{x'_n\}$, $x_n \in Q_0$ and $C_{v_\infty} \subset Q_1$. Again, as

C_{v_∞} is compact, we can pass to a subsequence to get $x'_n \rightarrow x_\infty$ where $x_\infty \in C_{v_\infty}$. In this case $x_n \rightarrow x_\infty$ also.

We note that Q is the Dasgupta–Hruska cut point tree \mathcal{T}_{cut} for $\mathcal{D}(\mathcal{L}_g)$. The above discussion shows that $\mathcal{D}(\mathcal{L}_g)$ is the *inverse limit of a tree of circles* [11, Section 4.1].

We will now show that quadratic elements in G are precisely those for which the decomposition space $\mathcal{D}(\mathcal{L}_g)$, or $\mathcal{D}_{(G_B, \mathcal{P}_B)}(\mathcal{L}_g^B)$ in the case of cut points as in Example 9.2, is homeomorphic to a circle. In the setting $\mathcal{A} = \emptyset$, this was originally shown by Otal [29, Theorem 2], see also the work of Cashen–Macura [8, Theorem 6.1]. First, we require one more definition due to Guirardel–Horbez [16, Lemma 2.19].

Definition 9.3. Let g be a non-peripheral element in G that is not simple and let $T \in \mathcal{O}$ be a Grushko tree. We say g is *quadratic in T* if its axis T_g intersects every orbit of edges in T exactly twice (regardless of orientation).

Note, the definition of *quadratic in T* by Guirardel–Horbez also requires that g is not conjugate to it inverse. As we are requiring that G is torsion-free, this is not possible. (Although, it can be shown using Whitehead graphs that Definition 9.3 implies that g is not conjugate to its inverse.) For a definition of the notion of an inverse limit of a tree of compacta see the work of Dasgupta–Hruska and the references within [11, Definition 4.3].

Proposition 9.4. *Let g be a non-peripheral element of (G, \mathcal{A}) that is not simple. The following are equivalent.*

- (1) *The element g is quadratic.*
- (2) *The decomposition space $\mathcal{D}(\mathcal{L}_g)$ is homeomorphic to an inverse limit of a tree of circles (possibly trivial).*
- (3) *For any Grushko tree $T \in \mathcal{O}$ that is Whitehead reduced for g and for any vertex $v \in T$, the Whitehead graph $\text{Wh}_T(\mathcal{L}_g, v)$ is a circle.*
- (4) *For any Grushko tree $T \in \mathcal{O}$ that is Whitehead reduced for g , g is quadratic in T .*

Proof. Let g be a non-peripheral element of (G, \mathcal{A}) that is not simple.

First we suppose that g is quadratic. Let Q be the (G, \mathcal{A}) -tree as in Definition 9.1 and let v be the vertex in Q where $\text{Stab}_Q(v) \cong \pi_1(\Sigma)$ for some compact, connected surface Σ .

There are two cases, depending on whether or not Q is trivial, i.e., a single vertex. If Q is trivial, then every boundary subgroup of Σ , other than the one generated by g , is a maximal peripheral subgroup. This implies that each of the peripheral factors in \mathcal{A} is cyclic and that G is a free group. The universal cover of Σ embeds in the hyperbolic plane and the quotient maps $\partial G \rightarrow \partial(G, \mathcal{A}) \rightarrow \mathcal{D}(\mathcal{L}_g)$ is the circular version of the Cantor map. See [7, Example 1.3] for more details. Hence $\mathcal{D}(\mathcal{L}_g)$ is homeomorphic to a circle, which trivially can be viewed as the inverse limit of a tree of circles.

Next, we assume that Q is non-trivial. It is easy to see that Q is a JSJ decomposition of $(G, \mathcal{A} \cup \mathcal{N}_g)$ over parabolic subgroups and thus without loss of generality, we can assume that it represents the canonical JSJ tree of cylinders for splittings of $(G, \mathcal{A} \cup \mathcal{N}_g)$ over peripheral subgroups (see [16, Remark 2.13]). Thus Q is the cut point tree constructed by Dasgupta–Hruska [11, Theorem 1.2(1)]. The vertex $v \in Q$ corresponds to a vertex $B \in V_0(\mathcal{T}_{\text{cut}})$. Let G_B , \mathcal{P}_B and \mathcal{L}_g^B be as in Lemma 8.3. By the first case when Q is trivial, we have that $\mathcal{D}_{(G_B, \mathcal{P}_B)}(\mathcal{L}_g^B)$ is a circle. Hence, as every vertex in $V_0(\mathcal{T}_{\text{cut}})$ is a translate of B , we have that $\mathcal{D}(\mathcal{L}_g)$ is homeomorphic to an inverse limit of a tree of circles by the result of Dasgupta–Hruska [11, Theorem 1.2(3)]. This shows that (1) implies (2).

Now suppose that $\mathcal{D}(\mathcal{L}_g)$ is homeomorphic to an inverse limit of a tree of circles. In particular, there are only two types of minimal cut sets in $\mathcal{D}(\mathcal{L}_g)$:

- (i) a cut point corresponding to a vertex in $V_1(\mathcal{T}_{\text{cut}})$; or
- (ii) a cut pair that belong to one of the circles comprising the inverse limit.

In particular, any minimal cut set that is not a single cut point consists of two points. By Lemmas 7.8 and 7.9, if $T \in \mathcal{O}$ is Whitehead reduced for g and $v \in T$ is a vertex, then every edge cut set in $\text{Wh}_T(\mathcal{L}_g, v)$ consists of exactly two edges. (Note $q(g^\infty)$ cannot be a cut point as g is not simple, c.f. proof of Lemma 8.2.) Since g is not simple, the Whitehead graph $\text{Wh}_T(\mathcal{L}_g, v)$ is connected and hence it is a circle. Thus (2) implies (3).

Suppose that $T \in \mathcal{O}$ is a Grushko tree such that for any vertex $v \in T$, the Whitehead graph $\text{Wh}_T(\mathcal{L}_g, v)$ is a circle. Thus every vertex in the Whitehead graph has degree equal to two. This implies that every edge incident to v is crossed by exactly two translates of T_g . As this holds for every vertex in T , we have that g is quadratic in T . Hence (3) implies (4).

Guirardel–Horbez prove that if g is quadratic in some Grushko tree $T \in \mathcal{O}$, then g is quadratic whenever g is not conjugate to its inverse [16, Lemma 2.19]. As explained before, as we are assuming that G is torsion-free, g cannot be conjugate to its inverse. Hence this result of Guirardel–Horbez gives us that (4) implies (1). \square

We record the following consequence of Lemma 8.3 and Proposition 9.4.

Corollary 9.5. *If a non-peripheral element $g \in G$ is \mathcal{Z} -simple but not simple nor quadratic, then $\mathcal{D}(\mathcal{L}_g)$ contains an inseparable loxodromic cut pair.*

Proof. Indeed, using the notation from Lemma 8.3, as g is not quadratic, we must have that $\mathcal{D}_{(G_B, \mathcal{P}_B)}(\mathcal{L}_g^B)$ contains an inseparable loxodromic cut pair by Lemma 8.3(5) and Proposition 9.4. By Lemma 8.3(6) this is also an inseparable cut pair in $\mathcal{D}(\mathcal{L}_g)$. \square

10. PROOF OF THEOREM 1.1 FOR QUADRATIC ELEMENTS

Using Proposition 9.4 and a corollary of Guirardel–Horbez, we can prove Theorem 1.1 for quadratic elements with a strategy that is similar to the one for simple elements in Section 5.

Proposition 10.1. *Let (G, \mathcal{A}) be a non-sporadic torsion-free free product. For all $L > 0$, there is a $D_1 > 0$ such that if $g \in G$ is quadratic, then the diameter of $\pi(\mathcal{O}_L(g)) \subset \mathcal{ZF}$ is at most D_1 .*

Proof. Given L , we set $D_1 = 2L + 5$.

Let $g \in G$ be a quadratic element and consider a Grushko tree $T_0 \in \mathcal{O}_L(g)$. As g is not simple, by Corollary 4.7 there is a Grushko tree $T \in \mathcal{O}_L(g)$ where $d(\pi(T_0), \pi(T)) \leq L$ that is Whitehead reduced for g . By Proposition 9.4, we have that g is quadratic in T . Guirardel–Horbez proved that in this case, there is a \mathcal{Z} -splitting $S_0 \in \mathcal{ZF}$ such that $d(\pi(T), S_0) \leq 2$ and in which g is elliptic [16, Corollary 2.20]. We have that $d(\pi(T_0), S_0) \leq L + 2$.

Given any other Grushko tree $T_1 \in \mathcal{O}_L(g)$, repeating the argument from above, we have that there is a \mathcal{Z} -splitting $S_1 \in \mathcal{ZF}$ in which g is elliptic and where $d(\pi(T_1), S_1) \leq L + 2$.

As g is elliptic in both S_0 and S_1 , we have that $d(S_0, S_1) \leq 1$. Therefore, we find:

$$d(\pi(T_0), \pi(T_1)) \leq d(\pi(T_0), S_0) + d(S_0, S_1) + d(S_1, \pi(T_1)) \leq 2L + 5 = D_1. \quad \square$$

11. FINDING BOUNDED LENGTH SPLITTING ELEMENTS

The main result of this section is Proposition 11.4. This proposition states that for a \mathcal{Z} -simple element $g \in G$ that is not simple nor quadratic, given a Grushko tree $T \in \mathcal{O}_L(g)$ that is Whitehead reduced for g , there is a \mathcal{Z} -splitting S where g is elliptic and an element a that fixes an edge in S whose length is bounded in terms of L . This is the main ingredient for proving Theorem 1.1 for \mathcal{Z} -simple elements, which is carried out in the next section in Proposition 12.1. The element a

essentially plays the role of a boundary curve for the subsurface filled by a given curve $\gamma \subset \Sigma$ as described in the “non-filling” case of Theorem 1.2 in the Introduction. The strategy for finding the element a is similar to that carried out by Cashen–Macura [8] and Cashen [7] who proved a similar statement in the case when $\mathcal{A} = \emptyset$ and thus G is free. Specifically, we seek to analyze cut pairs in the decomposition space $\mathcal{D}(\mathcal{L}_g)$.

Let (G, \mathcal{A}) be a fixed non-sporadic torsion-free free product. The next two lemmas will be used to control the number of components as we start to splice together Whitehead graphs.

Lemma 11.1. *Let \mathcal{L} be a periodic line collection. Suppose that $T \in \mathcal{O}$ is a Grushko tree that is Whitehead reduced for \mathcal{L} . Let v be a vertex in $V_\infty(T)$, fix two distinct vertices $Y, Y' \in T_v(\mathcal{L})$, and consider the subgraph $K \subset T_v(\mathcal{L})$ spanned by the directions at v other than Y and Y' . The following statements hold.*

- (1) *The vertex Y is adjacent in $T_v(\mathcal{L})$ to a vertex $Z \in K$ that belongs to an infinite component of K . Likewise, the vertex Y' is adjacent in $T_v(\mathcal{L})$ to a vertex $Z' \in K$ that belongs to an infinite component of K .*
- (2) *For each finite component $K_0 \subset K$, there are vertices $Z, Z' \in K_0$ such that Y is adjacent to Z and Y' is adjacent to Z' in $T_v(\mathcal{L})$.*

We note the possibility that $Z = Z'$ in the above statements.

Proof. We consider the embedded line $\alpha_Y: \mathbb{R} \rightarrow T_v(\mathcal{L})$ from Lemma 7.2. If $Y' \notin \alpha_Y([0, \infty))$ we set $Z = \alpha_Y(1)$. Since $\alpha_Y([1, \infty)) \subseteq K$ we see that Z belongs to an infinite component of H . If $Y' \notin \alpha_Y((-\infty, 0])$ we set $Z = \alpha_Y(-1)$ argue similarly. As $Y \neq Y'$ and α_Y is an embedding, we cannot have that both $Y' \in \alpha_Y((-\infty, 0])$ and $Y' \in \alpha_Y([0, \infty))$. Thus Z is well-defined. We can repeat the argument using $\alpha_{Y'}$ to find $Z' \in K$ that is adjacent to Y' and that lies in an infinite component of K . This proves (1).

Suppose that $K_0 \subset K$ is a finite component, fix a vertex $Y_0 \in K_0$ and consider the embedded line $\alpha_{Y_0}: \mathbb{R} \rightarrow T_v(\mathcal{L})$ from Lemma 7.2. As Y_0 is in a finite component of K , we must have a pair of integers $i < 0 < j$ where, after possibly precomposing α_{Y_0} by $t \mapsto -t$, $\alpha_{Y_0}(i) = Y$ and $\alpha_{Y_0}(j) = Y'$. We set $Z = \alpha_{Y_0}(i + 1)$ and $Z' = \alpha_{Y_0}(j - 1)$. Then Y is adjacent to Z and Y' is adjacent to Z' . Moreover, Z and Z' belong to K_0 as evidenced by the subpath of α_{Y_0} from $\alpha_{Y_0}(i + 1)$ to $\alpha_{Y_0}(j - 1)$. This proves (2). \square

Lemma 11.2. *Let \mathcal{L} be a periodic line collection. Suppose that $T \in \mathcal{O}$ is a Grushko tree that is Whitehead reduced for \mathcal{L} . Let $X \subset T$ be an arc whose endpoints are the midpoints of distinct edges e_- and e_+ , and let Y_- (respectively Y_+) denote the direction of $T - X$ that meets e_- (respectively Y_+). The following are true.*

- (1) *Each component of the graph $\text{Wh}_T(\mathcal{L}, X) - \{[Y_-], [Y_+]\}$ has a loose end both at $[Y_-]$ and at $[Y_+]$.*
- (2) *The number of components of the graph $\text{Wh}_T(\mathcal{L}, X) - \{[Y_-], [Y_+]\}$ is bounded above by $\#|\mathcal{L}|_T$.*

Proof. We prove both statements at the same time using induction on the number of vertices in T that belong to X via splicing finite stars.

Suppose that there is a single vertex v in T that belongs to X . If $\text{Stab}_T(v) = \{1\}$, then $\text{Wh}_T(\mathcal{L}, X)$ is isomorphic to $\text{Wh}_T(\mathcal{L}, v)$ as the inclusion $T - X \subset T - v$ induces a bijection on directions. As T is Whitehead reduced for \mathcal{L} , this graph does not have an admissible cut. Therefore it is connected and does not have a cut vertex. If some component U of $\text{Wh}_T(\mathcal{L}, X) - \{[Y_-], [Y_+]\}$ does not have a loose end at $[Y_-]$ nor at $[Y_+]$, then U is also a component of $\text{Wh}_T(\mathcal{L}, X)$ and therefore $\text{Wh}_T(\mathcal{L}, X)$ is not connected, which is a contradiction. Likewise, if some component of

$\text{Wh}_T(\mathcal{L}, X) - \{[Y_-], [Y_+]\}$ has a loose end at $[Y_-]$ but not at $[Y_+]$, then $[Y_-]$ is a cut vertex for $\text{Wh}_T(\mathcal{L}, X)$, which again is a contradiction. Similarly switching the roles of $[Y_-]$ and $[Y_+]$. As each component of $\text{Wh}_T(\mathcal{L}, X) - \{[Y_-], [Y_+]\}$ has a loose end at $[Y_-]$, the number of components in this graph is bounded above by the valence of the vertex $[Y_-]$ in $\text{Wh}_T(\mathcal{L}, X)$. This quantity is bounded above by the number of lines $\ell \in \mathcal{L}$ such that ℓ_T meets e_- , which is bounded above by $|\mathcal{L}|_T$. This verifies the base cases for both statements when $\text{Stab}_T(v) = \{1\}$.

If $\text{Stab}_T(v) = \{\infty\}$ then we consider the graph $T_v(\mathcal{L})$ and apply Lemma 11.1. Both Y_- and Y_+ represent vertices of $T_v(\mathcal{L})$ and the graph $T_v(\mathcal{L}) - X$ is the subgraph spanned by the complement of these two vertices. By Lemma 11.1(1), the vertex $\hat{v} \in V_X(\mathcal{L})$ representing every infinite component of $T_v(\mathcal{L}) - X$ has an edge in $\text{Wh}_T(\mathcal{L}, v)$ connecting it to $[Y_-]$ and an edge in $\text{Wh}_T(\mathcal{L}, v)$ connecting it to $[Y_+]$. Any other vertex Z in $V_X(\mathcal{L})$ is a vertex of $T_v(\mathcal{L}) - X$ that belongs to a finite component. Hence, by Lemma 11.1(2), there is an edge path in $\text{Wh}_T(\mathcal{L}, v)$ connecting $[Z]$ to $[Y_-]$ and an edge path in $\text{Wh}_T(\mathcal{L}, v)$ connecting $[Z]$ to $[Y_+]$. As before, this shows that the number of components is bounded above by the valence of Y_- in $T_v(\mathcal{L})$, which again as before, is bounded by $\#|\mathcal{L}|_T$. This completes the verification of the base case.

For the inductive step, we just observe that splicing together two graphs with loose ends where every component in each has a loose end at each missing vertex results in a graph with loose ends that also has this property and that the number of components does not increase. \square

The following proposition shows that every inseparable loxodromic cut pair is given by an element with bounded length. The argument below is modeled off of that by Cashen [7, Lemma 3.26 & Proposition 4.13] (see also [8, Lemma 4.14]). A version of this argument for hyperbolic groups was given by Barrett [1, Section 2.3].

Proposition 11.3. *For all $L > 0$, there is an $R_0 > 0$ with the following property. If g is a non-peripheral element of G , $T \in \mathcal{O}_L(g)$ is a Grushko tree that is Whitehead reduced for g , and $\mathcal{D}(\mathcal{L}_g)$ contains an inseparable loxodromic cut pair $\{q(h^\infty), q(h^{-\infty})\}$, then there exists a non-peripheral element $a \in G$ where $|a|_T \leq R_0$ and $T_a = T_h$.*

Proof. Fix a non-peripheral element $g \in G$ and suppose that $T \in \mathcal{O}_L(g)$ a Grushko tree that is Whitehead reduced for g . By Proposition 6.3, we have that $\mathcal{D}(\mathcal{L}_g)$ is connected. As $\{q(h^\infty), q(h^{-\infty})\}$ is a cut pair, by Lemma 7.4 we have that $\text{Wh}_T(\mathcal{L}_g, T_h)$ is not connected. Enumerate the components of $\text{Wh}_T(\mathcal{L}_g, T_h)$ by $1, \dots, c$ and observe that $c \leq L$ by Lemma 11.2(2) since h acts cocompactly on T_h . Set $R_0 = 2\xi(G, \mathcal{A})c^L + 1$. Fix a segment $X_0 \subset T_h$ of length R_0 . There is some edge $e \subset X_0$ whose orbit meets X_0 with the same orientation at least $c^L + 1$ times.

Let \mathcal{L}_e be the subset of lines $\ell \in \mathcal{L}_g$ such that ℓ_T contains the edge e . We observe that each line in \mathcal{L}_e corresponds to an edge in $\text{Wh}_T(\mathcal{L}, T_h)$. The set \mathcal{L}_e contains L_e elements where $L_e \leq L$. Fix a bijection $\beta: \{1, \dots, L_e\} \rightarrow \mathcal{L}_e$. Consider the function $f: \{1, \dots, L_e\} \rightarrow \{1, \dots, c\}$ where $f(i)$ records the component of $\text{Wh}_T(\mathcal{L}_g, T_h)$ that the edge associated to $\beta(i)$ belongs to.

If $a' \in G$ is such that $a'e$ meets X_0 with the preferred orientation, then there is an associated bijection $\beta_{a'}: \mathcal{L}_e \rightarrow \mathcal{L}_{a'e}$ given by $\beta_{a'}(\ell) = a'\ell$. We get another function $f_{a'}: \{1, \dots, L_e\} \rightarrow \{1, \dots, c\}$ using the bijection $\beta_{a'} \circ \beta$. In other words, the function $f_{a'}(i)$ records the component of $\text{Wh}_T(\mathcal{L}_g, T_h)$ that the edge associated to $a'\beta(i)$ belongs to.

As the orbit of e meets X_0 with the same orientation at least $c^L + 1$ times, this procedure produces $c^L + 1$ functions $f_{a'}: \{1, \dots, L_e\} \rightarrow \{1, \dots, c\}$. Hence two must be identical functions. Let $a_0, a_1 \in G$ be two such elements where $f_{a_0} = f_{a_1}$ and set $a = a_1 a_0^{-1}$. We observe that the edges $a_0 e$ and $a_1 e$ belong to T_a as they are coherently oriented. In particular, a is non-peripheral and $|a|_T \leq R_0$.

Let x be the midpoint of a_0e and let $X \subset T$ be the arc from x to ax . Let $Y_- \subset T$ be the direction of $T - X$ that meets a_0e and let $Y_+ \subset T$ be the direction of $T - X$ that meets a_1e . Splicing the translates of $\text{Wh}_T(\mathcal{L}_g, X) - \{[Y_-], [Y_+]\}$ by powers of a together to form $\text{Wh}_T(\mathcal{L}_g, T_a)$ may introduce some additional connections between the components. However, by the construction of a , we will never splice together components that lie in separate components of $\text{Wh}_T(\mathcal{L}_g, T_h)$. Hence, we see that the number of components of $\text{Wh}_T(\mathcal{L}_g, T_a)$ is bounded below by c . Thus $\{q(a^\infty), q(a^{-\infty})\}$ is a loxodromic cut pair. It remains to show that $T_a = T_h$, in other words, a and h are powers of the same element in G .

Without loss of generality, we assume that a and h translate in the same direction along $T_a \cap T_h$ which necessarily includes the segment X . Suppose that $T_a \neq T_h$. Then $T_a \cup T_h - (T_a \cap T_h)$ decomposes into four subsets $X_{a,+}, X_{a,-} \subset T_a$ and $X_{h,+}, X_{h,-} \subset T_h$ where $aX_{a,+} \subset X_{a,+}$, $a^{-1}X_{a,-} \subset X_{a,-}$, $hX_{h,+} \subset X_{h,+}$, and $h^{-1}X_{h,-} \subset X_{h,-}$. See Figure 11.

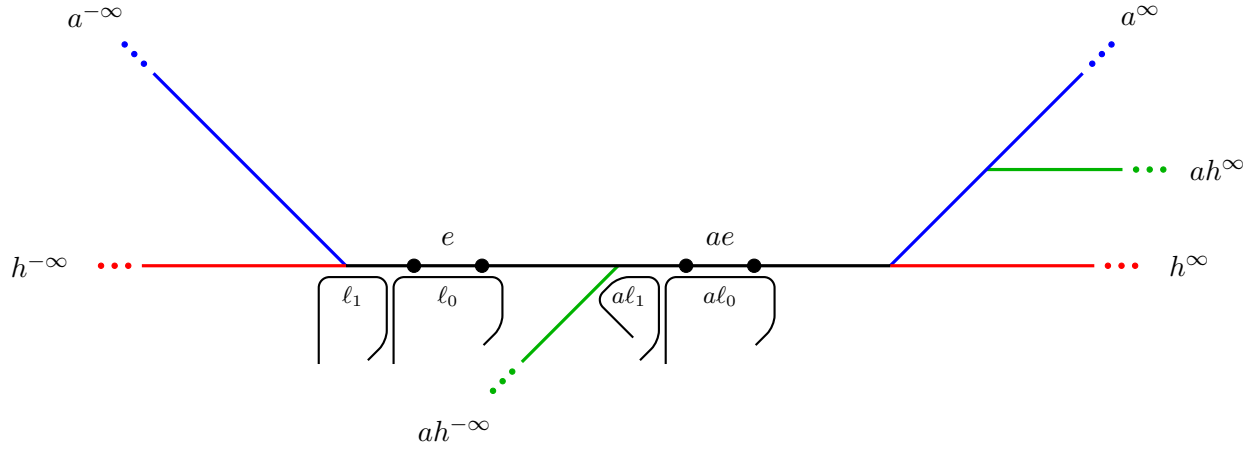


FIGURE 11. The set-up and proof of Proposition 11.3. The sets $X_{a,\pm}$ are shown in blue, the sets $X_{h,\pm}$ are shown in red and the set $T_a \cap T_h$ is shown in black.

Let $Y_{a,+}$ be the direction of $T - T_h$ that contains $X_{a,+}$ and similarly define $Y_{a,-}$ as the direction of $T - T_h$ that contains $X_{a,-}$. Since $\{q(a^{-\infty}), q(a^\infty)\}$ is a cut pair and as $\{q(h^{-\infty}), q(h^\infty)\}$ is an inseparable cut pair, the vertices $[Y_{a,-}]$ and $[Y_{a,+}]$ belong to the same component $C \subset \text{Wh}_T(\mathcal{L}_g, T_h)$. Let ℓ be a geodesic that crosses the edge e so that the edge of $\text{Wh}_T(\mathcal{L}_g, T_h)$ corresponding to ℓ does not belong to C . Let C' be the component $\text{Wh}_T(\mathcal{L}_g, T_h)$ that contains the edge corresponding to ℓ .

Claim 1. The points $q(ah^{-\infty})$ and $q(a^\infty)$ belong to different components of $\mathcal{D}(\mathcal{L}_g) - \{q(h^{-\infty}), q(h^\infty)\}$.

Proof of Claim 1. There is a sequence of geodesics $\ell = \ell_0, \dots, \ell_m$ in \mathcal{L}_g such that ℓ_m meets $X_{h,-}$ and where the edges corresponding to ℓ_{j-1} and ℓ_j in $\text{Wh}_T(\mathcal{L}_g, T_h)$ are incident on a common vertex for $j = 1, \dots, m$. By the choice of a , the edge corresponding to $a\ell$ belongs to C' and the sequence of lines $a\ell = a\ell_0, \dots, \ell_m$ show that the edge corresponding to $a\ell_m$ also belongs to C' . The direction $aX_{h,-} \subset T - T_h$ meets $a\ell_m$ and thus the vertex $[aX_{h,-}]$ also belongs to C' . This verifies the claim. \square

Claim 2. The points $q(ah^\infty)$ and $q(a^\infty)$ belong to the same components of $\mathcal{D}(\mathcal{L}_g) - \{q(h^{-\infty}), q(h^\infty)\}$.

Proof of Claim 2. This follows as the component $aX_{h,+}$ is contained in $X_{a,+}$. \square

Claim 1 and Claim 2 imply that $\{q(h^{-\infty}), q(h^{\infty})\}$ is separated by the cut pair $\{q(ah^{-\infty}), q(ah^{\infty})\}$. This is a contradiction as $\{q(h^{-\infty}), q(h^{\infty})\}$ is inseparable. Hence $T_a = T_h$ as desired. \square

We will now prove the main result of this section.

Proposition 11.4. *For all $L > 0$, there is an $R > 0$ with the following property. If $g \in G$ is \mathcal{Z} -simple but not simple nor quadratic and $T \in \mathcal{O}_L(g)$ is a Grushko tree that is Whitehead reduced for g , then there exists a \mathcal{Z} -splitting S in which g is elliptic and a non-peripheral element $a \in G$ with $|a|_T \leq R$ and where a fixes an edge in S .*

Proof. Let $g \in G$ be a non-peripheral \mathcal{Z} -simple element that is not simple nor quadratic and fix a tree $T \in \mathcal{O}_L(g)$ that is Whitehead reduced for g . By Corollary 9.5, the decomposition space $\mathcal{D}(\mathcal{L}_g)$ contains an inseparable loxodromic cut pair.

Let R be the constant R_0 from Proposition 11.3 using L . By Proposition 11.3, there is a non-peripheral element $a \in G$ where $|a|_T \leq R$ such that $\{q(a^{\infty}), q(a^{-\infty})\} \in \mathcal{D}(\mathcal{L}_g)$ is an inseparable loxodromic cut pair. Thus the image of this cut pair in $\mathcal{D}(\mathcal{L}_g \cup \mathcal{L}_a)$ is a cut point, denote it by x . Let \mathcal{T}_{cut} be the cut point tree for $\mathcal{D}(\mathcal{L}_g \cup \mathcal{L}_a)$ and let $v \in V_1(\mathcal{T}_{\text{cut}})$ be the vertex corresponding x . As seen in the proof of Proposition 11.3, the action of a on $\mathcal{D}(\mathcal{L}_g) - \{q(a^{\infty}), q(a^{-\infty})\}$ preserves the components, and hence a stabilizes an edge e in \mathcal{T}_{cut} that is incident to x . Collapsing all edges of \mathcal{T}_{cut} that do not lie in the orbit of e results in the desired \mathcal{Z} -splitting S . \square

12. PROOF OF THEOREM 1.1 FOR \mathcal{Z} -SIMPLE ELEMENTS

We can now prove Theorem 1.1 for \mathcal{Z} -simple elements. As for simple elements (Proposition 5.1) and quadratic elements (Proposition 10.1), the strategy is that given $T \in \mathcal{O}_L(g)$, we seek to find a \mathcal{Z} -splitting in which g is elliptic whose distance to $\pi(T)$ is bounded in terms of L . To do so, we first find a \mathcal{Z} -splitting in which g is elliptic and where some edge stabilizer has length bounded in terms of L using the work in Section 11.

Proposition 12.1. *Let (G, \mathcal{A}) be a non-sporadic torsion-free free product. For all $L > 0$, there is a $D_2 > 0$ such that if $g \in G$ is \mathcal{Z} -simple then the diameter of $\pi(\mathcal{O}_L(g))$ is at most D_2 .*

Proof. Let R be the constant R from Proposition 11.4 for L and set $D_2 = 2L + 2R + 5$.

By Proposition 5.1, if g is simple then the diameter of $\pi(\mathcal{O}_L(g))$ is at most $D_0 = 2L + 3 \leq D_2$.

By Proposition 10.1, if g is quadratic then the diameter of $\pi(\mathcal{O}_L(g))$ is at most $D_1 = 2L + 5 \leq D_2$.

Lastly, we suppose that g is not simple nor quadratic and consider a Grushko tree $T_0 \in \mathcal{O}_L(g)$. As g is not simple, by Corollary 4.7 there is a Grushko tree $T \in \mathcal{O}_L(g)$ where $d(\pi(T_0), \pi(T)) \leq L$ that is Whitehead reduced for g . By Proposition 11.4, there is a \mathcal{Z} -splitting $S_0 \in \mathcal{ZF}$ in which g is elliptic and an element $a \in G$ where $|a|_T \leq R$ and where a fixes an edge in S_0 . By Lemma 2.2, the element a is simple. As shown in the proof of Proposition 5.1, since a is simple, there is a \mathcal{Z} -splitting $S \in \mathcal{ZF}$ where a has a fixed point and such that $d(\pi(T), S) \leq R + 1$. Since a has a fixed point in both S and S_0 , we have $d(S, S_0) \leq 1$. Hence

$$d(\pi(T_0), S) \leq d(\pi(T_0), \pi(T)) + d(\pi(T), S) + d(S, S_0) \leq L + R + 2.$$

Repeating this argument for another tree $T_1 \in \mathcal{O}_L(g)$, we find another \mathcal{Z} -splitting $S_1 \in \mathcal{ZF}$ in which g is elliptic and such that $d(\pi(T_1), S_1) \leq L + R + 2$. Since g is elliptic in both S_0 and S_1 , we have $d(S_0, S_1) \leq 1$, and hence

$$d(\pi(T_0), \pi(T_1)) \leq d(\pi(T_0), S_0) + d(S_0, S_1) + d(S_1, \pi(T_1)) \leq 2L + 2R + 5. \quad \square$$

13. PROOF OF THEOREM 1.1

To complete the proof of Theorem 1.1 we use contradiction. Supposing that for some fixed L the diameter of $\pi(\mathcal{O}_L(g_n))$ is unbounded for some sequence (g_n) of non-peripheral elements, we will find a \mathcal{Z} -simple element g where the diameter of $\pi(\mathcal{O}_{L'}(g))$ is infinite for some L' . This is a contradiction by Proposition 12.1.

In order to obtain this contradiction, we must work with the closure $\overline{\mathcal{O}}$ of the deformation space and we recall this space now. For more information, see the work of Guirardel–Levitt [17] or Horbez [23]. Given an Grushko tree $T \in \mathcal{O}$ and a non-peripheral element $g \in G$, by $\|g\|_T$ we denote the translation length of g , that is $\|g\|_T = d_T(x, gx)$ for any $x \in T_g$. Thus each tree determines a *length function* $\|\cdot\|_T: G \rightarrow \mathbb{R}$. Culler–Morgan proved that the assignment $T \mapsto \|\cdot\|_T$ defines an injective function $\mathcal{O} \rightarrow \mathbb{R}^G$ [9]. The closure of \mathcal{O} in \mathbb{R}^G is denoted $\overline{\mathcal{O}}$. Horbez proved that $\overline{\mathcal{O}}$ is projectively compact and identified this closure with a space of action of G on \mathbb{R} -trees [23, Proposition 2.3]. Specifically, given a sequence $(T_n) \subset \overline{\mathcal{O}}$, after passing to a subsequence, there are real numbers (λ_n) such that the length functions $\frac{1}{\lambda_n} \|\cdot\|_{T_n}$ converge to the length function $\|\cdot\|_T$ of an \mathbb{R} -tree equipped with an action G [23, Theorem 1]. We observe that if $\|g\|_{T_n} \rightarrow \infty$ for some g , then necessarily we have that $\lambda_n \rightarrow \infty$.

We repeat the statement of Theorem 1.1 for the convenience of the reader.

Theorem 1.1. *Let (G, \mathcal{A}) be a non-sporadic torsion-free free product. For all $L > 0$, there is a $D > 0$ such that for any non-peripheral element $g \in G$, the diameter of $\pi(\mathcal{O}_L(g)) \subset \mathcal{ZF}$ is at most D .*

Proof. Suppose that the theorem is false. Then there is an L such that for all $n \geq 0$ there is a non-peripheral element $g_n \in G$ for which the diameter of $\pi(\mathcal{O}_L(g_n))$ is greater than n . We fix Grushko trees $S_n, T_n \in \mathcal{O}_L(g_n)$ with $d(\pi(S_n), \pi(T_n)) > n$. As scaling the length of edges does not change $|g_n|_{S_n}$ nor the image $\pi(S_n)$, we can scale all of the edges of S_n to have length one so that $|g_n|_{S_n} = \|g_n\|_{S_n}$. We similarly scale the edges on each T_n . After passing to subsequences and possibly replacing g_n , S_n and T_n respectively by $\theta_n^{-1}(g_n)$, $S_n \theta_n$ and $T_n \theta_n$ for some $\theta_n \in \text{Out}(G, \mathcal{A})$, we can assume that (S_n) is a constant sequence, $S_n = S$. With S and these new sequences (g_n) and (T_n) , we have that $\|g_n\|_S, \|g_n\|_{T_n} \leq L$ and $d(\pi(S), \pi(T_n)) \rightarrow \infty$.

If $h \in G$ is simple, then we must have that $\|h\|_{T_n} \rightarrow \infty$ by Proposition 5.1 as $d(\pi(S), \pi(T_n)) \rightarrow \infty$. Therefore, as described in the beginning of this section, after passing to subsequences, we have a sequence (λ_n) of positive real numbers and an \mathbb{R} -tree $T \in \overline{\mathcal{O}}$ where $\frac{1}{\lambda_n} \|\cdot\|_{T_n} \rightarrow \|\cdot\|_T$ and for which $\lambda_n \rightarrow \infty$.

Next, by passing to subsequences and replacing each element in the sequence (g_n) by a conjugate if necessary, we can assume that there is an edge e in S that is contained in each of the axes S_{g_n} . As $\partial(G, \mathcal{A})$ is compact, after passing to a subsequence, we have that $g_n^\infty \rightarrow \alpha \in \partial(G, \mathcal{A})$ and $g_n^{-\infty} \rightarrow \omega \in \partial(G, \mathcal{A})$. As the axes S_{g_n} all contain the edge e , we must have that $\alpha \neq \omega$.

Claim. There a non-peripheral element $g \in G$ for which $\|g\|_{T_n}$ stays bounded.

Proof of Claim. As the axes S_{g_n} are periodic and $|g_n|_S$ is bounded, we either have that $\alpha, \omega \in \partial_\infty(G, \mathcal{A})$ or that $\alpha, \omega \in V_\infty(G, \mathcal{A})$. We deal with these cases one at a time.

First, assume that $\alpha, \omega \in \partial_\infty(G, \mathcal{A})$. Thus, the axes of the elements g_n overlap in longer and longer segments in S . As the translation lengths are bounded, this means that the sequence is eventually constant, hence $g_n = g$ for some $g \in G$ and large enough n . In this case, this is our non-peripheral element for which $\|g\|_{T_n}$ stays bounded.

Otherwise, we have that $\alpha, \omega \in V_\infty(G, \mathcal{A})$. Let v and w be the distinct vertices of S that realize α and ω . Fix nontrivial elements $h_0 \in \text{Stab}_S(v)$ and $h_1 \in \text{Stab}_S(w)$ and set $g = h_0 h_1$. Now in T_n

the distance between v_n and w_n , the realizations of α and ω , stays bounded as $\|g_n\|_{T_n}$ is bounded. This means that the translation length $\|g\|_{T_n}$ is bounded as well (it is twice the distance between v_n and w_n). In this case, this is our non-peripheral element for which $\|g\|_{T_n}$ stays bounded. \square

To conclude, we observe that as $\lambda_n \rightarrow \infty$, since $\|g\|_{T_n}$ stays bounded, we have that g is elliptic in T as $\|g\|_T = \lim_{n \rightarrow \infty} \frac{\|g\|_{T_n}}{\lambda_n}$. It then follows from work of Guirardel–Levitt that g is \mathcal{Z} -simple [18, Corollary 9.10]. However, boundedness of $\|g\|_{T_n} = |g|_{T_n}$ implies that the diameter of $\pi(\mathcal{O}_{L'}(g))$ is infinite for some L' . This is a contradiction of Proposition 12.1. \square

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DEPARTMENT OF MATHEMATICAL SCIENCES, UNIVERSITY OF ARKANSAS, FAYETTEVILLE, AR 72701

Email address: mattclay@uark.edu

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF WISCONSIN, MADISON, 480 LINCOLN DRIVE, MADISON, WI 53706

Email address: caglar@math.wisc.edu